Optimality of payoffs in Lévy models

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Abstract

In this paper we determine the lowest cost strategy for a given payoff in general Lévy markets where the pricing is based on the Esscher martingale measure. In particular, we consider Lévy models where the price process is driven by an NIG- and a VG-process. Explicit solutions for cost-efficient strategies are derived for a variety of vanilla options, spreads, and forwards. Calculations of efficient put prices based on estimated parameters from German stock prices show that the potential savings the optimal payoffs provide can be quite substantial. The empirical findings are supplemented by a theoretical result that relates the magnitude of these savings to the strength of the market trend. Moreover, we consider the problem of hedging such claims, derive explicit formulas for the deltas of efficient calls and puts and apply the results to German stock market data. As a main result we find that cost-efficient options also show an improved behaviour concerning delta hedging compared to their classical counterparts.

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1 Introduction

In this paper we study optimal investment decisions in incomplete markets where the prices of risky assets are driven by Lévy processes. In particular, we solve for the investment strategy with minimal costs that achieves a given

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payoff-distribution. This strategy is called cost-efficient with respect to the given distribution.

Finding efficient strategies for a given probability distribution is a classical task which has undergone a steady development and is closely related to attaining the Fréchet bounds of multivariate random variables which were developed by Fréchet in 1935 (see also Fréchet (1951)). Since then these bounds were applied in a variety of related fields as, for instance, financial economics, insurance mathematics, risk management, statistics, decision theory and control and optimization theory. Improved Fréchet bounds were derived in Rachev and Rüschendorf (1994) by including additional information on the dependence structure.

Our work builds on a stream of related literature. In an arbitrage-free and complete market setting Dybvig (1988a,b) characterizes efficient payoffs for rational agents with increasing preferences who care only about the distribution of terminal wealth. Moreover, he extensively discusses a way to construct cost-efficient payoffs for a given distribution function. Markets which include frictions such as transaction costs have been studied by Pelsser and Vorst (1996) and Jouini and Kallal (2001). Bernard et al. (2012) propose two alternative approaches to strictly improve an inefficient financial strategy and present an explicit representation of the cost-efficient strategy to achieve a given payoff-distribution.[1] This representation is closely linked to the concept of countermonotonicity. They construct financial derivatives that dominate the original payoff in the sense of first- and second-order stochastic dominance in the Black–Scholes setting.

The major contribution in the first part of our paper (Sections 2 and 3) is the application and extension of the theoretical results of Bernard et al. (2012) to a more general Lévy market setting. For this purpose we consider, among others, Variance Gamma (VG) and normal inverse Gaussian (NIG) processes as drivers for the prices of risky assets. In comparison to the geometric Brownian motion underlying the Black–Scholes model these Lévy processes provide more precision and flexibility.[2] We further assume an arbitrage-free and frictionless market where trading takes place continuously, and we suppose that there exists a constant risk-free interest rate. However, since Lévy markets are incomplete, there does not exist a unique state-price density. Therefore, we hypothesize that all agents in the market agree on the Esscher martingale measure for pricing. For a variety of relevant financial derivatives we explicitly derive cost-efficient strategies, that is, we improve the payoffs in the sense of the stochastic order $\leq_{\mathrm{st}}$ for agents

[1] A previous version of this paper is Bernard and Boyle (2010).

[2] For more information about their applicability in finance we refer to Raible (2000) and Schoutens (2003).

with increasing preferences. In this paper we confine ourselves to studying path-independent derivatives only, but this is not an essential restriction: Vanduffel et al. (2009) have shown that for general Lévy markets where the arbitrage-free pricing is based on the Esscher transform, path-dependent payoffs are inefficient with respect to the convex stochastic order $\leq_{\text{cx}}$ and can be improved by conditioning on the price of the underlying at maturity. But these enhanced payoffs then are path-independent and thus can be optimized further using the techniques described in this paper.

We also investigate the impact of the market behaviour on the cost reduction that can be achieved by investing in the efficient strategies. Roughly speaking, the overall behaviour (bullish or bearish) is characterized by the sign of the risk-neutral Esscher parameter, and the size of its absolute value determines the strength of the market trend. We show that the more pronounced the trend, the higher are the price differences between inefficient and optimal strategies. Moreover, we formally prove the intuitive result that the cost-efficient strategy corresponding to a long position of a certain strategy coincides with a short position in the most-expensive strategy corresponding to a short position in the original strategy. We further show that a cost-efficient version of the put-call parity exists stating that the cost-efficient strategy corresponding to a portfolio of a long call and a short put agrees with the cost-efficient strategy for a long forward.

The main contribution of the second part of the paper (Section 4) is the explicit derivation of hedging strategies for cost-efficient payoffs. Specifically, we provide formulas to compute Greek delta, that is, the derivative of the cost of a strategy with respect to the underlying, for cost-efficient strategies corresponding to European call and put options. This is particularly important for practical applications as the pricing formulas for cost-efficient strategies themselves are still unsatisfying if no hedging strategies exist. Moreover, we prove that the magnitude of the deltas of efficient calls and puts in almost all cases is smaller than that of the deltas of the corresponding vanilla options. This suggests that also the hedging errors arising in discrete delta hedging strategies should be smaller for cost-efficient options than for standard options. In a practical application using German stock price data we demonstrate that this is indeed the case: The accumulated absolute hedge errors obtained from delta hedging of vanilla puts on two German stocks are always higher than those of efficient puts. This shows that cost-efficient options not only provide a cheaper way of realizing a certain payoff, but can also be hedged more accurately.

The paper is structured as follows: In Section 2, we present a generalization of Proposition 4 in Bernard et al. (2012), give an explicit characterization of cost-efficient strategies in Lévy markets when pricing is based on the

Esscher martingale measure, explain how the magnitude of the price difference between standard and efficient options is influenced by the risk-neutral Esscher parameter, and also describe the Lévy market models. Section 3 contains the derivations of cost-efficient strategies for European puts and calls, spreads, and forwards as well as a comparison of efficient and vanilla put prices which are calculated using estimated parameters from German stock prices. Moreover, we discuss the put-call parity and the relation between cost-efficient and most-expensive strategies here. In Section 4, we present formulas for the Greek delta of cost-efficient call and put options, show that they have a smaller size than the delta of a vanilla call resp. put, and apply the theoretical results to explicitly compute and visualize the hedge errors arising in delta hedging of efficient and vanilla puts on two German stocks. In Section 5 we state some conclusions. An appendix contains detailed derivations of the risk-neutral Esscher parameters for the different Lévy models considered in the paper.

2 Cost-efficient strategies

2.1 General concepts and definitions

Consider a financial market on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ satisfying the usual conditions. We assume that the financial market is incomplete, but free of arbitrage, perfectly liquid and frictionless. Let $(S_t)_{t\geq 0}$ denote the price process of a risky asset and let r be the constant deterministic risk-free interest rate. Further, we assume that all agents in the market agree on the same state-price density $(Z_t)_{t\geq 0}$ for pricing, where the process $(Z_t)_{t\geq 0}$ is chosen such that the discounted process $(e^{-rt}Z_tS_t)_{t\geq 0}$ is a P-martingale. In other words, Z_t equals the Radon-Nikodym derivative $\frac{dQ}{dP}\big|_{\mathcal{F}_t}$ of the risk-neutral measure Q with respect to the real-world measure P. We assume this general setup throughout the whole paper and indicate whenever we concretize it.

In this paper we are interested in strategies of European type yielding a given terminal payoff-distribution, and among those, especially in the ones with minimal and maximal costs. Here the cost of a strategy with a given terminal payoff is defined as the discounted expected payoff w.r.t. the state-price density Z_T. A strategy is called *cost-efficient* if it has minimal costs among all strategies producing the same given payoff-distribution. Formally, we have the following definitions.

Definition 2.1 (Cost of a strategy) The cost of a strategy with terminal payoff X_T, $T > 0$, is given by

$$c(X_T) = E\big[e^{-rT} Z_T X_T\big],$$

provided that the expectation exists.

Note that here and in the following the expectation $E[\cdot] = E_P[\cdot]$ is always calculated with respect to the real-world measure P if not stated otherwise.

Definition 2.2 (Cost-efficient and most-expensive strategies)

a) A strategy (or payoff) $\underline{X}_T \sim G$ is called *cost-efficient* w.r.t. the payoff-distribution G if any other strategy X_T that generates the same payoff-distribution G costs at least as much, that is,

$$c(\underline{X}_T) = E[e^{-rT} Z_T \underline{X}_T] = \min_{\{X_T \sim G\}} E[e^{-rT} Z_T X_T].$$

b) A strategy (or payoff) $\overline{X}_T \sim G$ is called *most-expensive* w.r.t. the payoff-distribution G if any other strategy X_T that generates the same payoff-distribution G costs at most as much, that is,

$$c(\overline{X}_T) = E[e^{-rT} Z_T \overline{X}_T] = \max_{\{X_T \sim G\}} E[e^{-rT} Z_T X_T].$$

c) The *efficiency loss* of a strategy with payoff $X_T \sim G$ at maturity T is defined as

$$c(X_T) - c(\underline{X}_T).$$

It has recently be shown in Bernard et al. (2012) that cost-efficiency is closely related to the theory of stochastic orders and dependence concepts. In particular, the net profit from investing into the cost-efficient strategy $\underline{X}_T$ is always greater than that of X_T in the stochastic order $\leq_{\text{st}}$: Recall that a random variable X is said to be smaller than Y with respect to the stochastic order ($X \leq_{\text{st}} Y$) if $F_X(x) \geq F_Y(x)$ or, equivalently, $P(X > x) \leq P(Y > x)$ for all $x \in \mathbb{R}$. Thus, if X and Y are two possible investment strategies with $X \leq_{\text{st}} Y$, a speculative investor who is just interested in maximizing the gain will always prefer Y to X because Y yields higher outcomes with greater probabilities (and lower ones with smaller probabilities). Since the payoff X_T is realized at time T, the net profit from investing in X_T at $t = 0$ instead of saving the initial costs $c(X_T)$ is $X_T - c(X_T)e^{rT}$. By Definition 2.2, the payoffs X_T and $\underline{X}_T$ have the same distribution function G, but $c(\underline{X}_T) \leq c(X_T)$, hence

$$\begin{aligned} P\big(\underline{X}_T - c(\underline{X}_T)e^{rT} \leq x\big) &= G\big(x + c(\underline{X}_T)e^{rT}\big) \\ &\leq G\big(x + c(X_T)e^{rT}\big) = P\big(X_T - c(X_T)e^{rT} \leq x\big) \end{aligned}$$

which means that $X_T - c(X_T)e^{rT} \leq_{\text{st}} \underline{X}_T - c(\underline{X}_T)e^{rT}$. Thus, as intuitively expected, speculators will always prefer the cost-efficient strategies.

Bernard et al. (2012) provide explicit examples of cost-efficient strategies for the case of a complete Black–Scholes market. In this section we reformulate their theoretical results for general Lévy markets where the state-price density is given by an Esscher transform. Therefore, in a first step we characterize cost-efficient strategies in general incomplete markets by means of the lower and upper Fréchet bounds corresponding to counter- resp. comonotonic pairs of random variables. Afterwards, we explicitly compute these bounds in Lévy market models where the Esscher martingale measure is used for pricing.

Recall that for any two given marginal distributions F_1 and F_2 the Fréchet class $\mathcal{M}(F_1, F_2)$ is the set of all bivariate distributions F having the marginals F_1, F_2. For any distribution $F \in \mathcal{M}(F_1, F_2)$, there are sharp upper and lower bounds, called *Fréchet bounds*: For all $x = (x_1, x_2)^\top \in \mathbb{R}^2$ it holds that

$$F^-(x) \leq F(x) \leq F^+(x) \tag{2.1}$$

where

$$\begin{aligned} F^-(x) &= \max\{F_1(x_1) + F_2(x_2) - 1, 0\}, \\ F^+(x) &= \min\{F_1(x_1), F_2(x_2)\}. \end{aligned} \tag{2.2}$$

For bivariate distribution functions, these bounds trace back to Fréchet (1935). The upper and lower bounds for $d \geq 3$ dimensions were described in Dall'Aglio (1972). Sharpness of the lower bound in the case $d \geq 3$ was first proven in Rüschendorf (1981).

In the bivariate case, $F^+(x)$ and $F^-(x)$ are themselves distribution functions. As we shall see below, pairs of random variables that follow either of these distributions exhibit very strong dependencies which are formally defined by

Definition 2.3 Let $X = (X_1, X_2) \sim F$ be a bivariate random vector with joint distribution function F and marginals F_1, F_2.

i) X_1 and X_2 are said to be *comonotonic* if $F = F^+$.

ii) X_1 and X_2 are said to be *countermonotonic* if $F = F^-$.

Note that the above definition implies that co- resp. countermonotonic random vectors are unique (in distribution) once the marginal distributions are fixed. Moreover, one can easily construct random pairs (X_1, X_2) with this property as follows: Let $U \sim U(0, 1)$ be a random variable that is uniformly distributed on $(0, 1)$, and let F_i^{-1} denote the *generalized inverse*

$$F_i^{-1}(u_i) = \inf\{y \mid F_i(y) \geq u_i\}, \quad u_i \in (0, 1),$$

of the univariate distribution function F_i, $i = 1, 2$, then the random vector

$$(X_1, X_2) = (F_1^{-1}(U), F_2^{-1}(U))$$

is comonotonic and has the joint distribution function $F^+(x)$. Similarly,

$$(X_1, X_2) = (F_1^{-1}(U), F_2^{-1}(1-U))$$

is countermonotonic with joint distribution function $F^-(x)$.

Under the additional assumption that at least one distribution function F_i is continuous, one can deduce from the above representations that X_1 and X_2 are countermonotonic if and only if $X_j = h(X_i)$ a.s. for some decreasing function h and $i, j \in \{1, 2\}, i \neq j$ (suppose that F_1 is continuous, then one can set $U = F_1(X_1)$ and $h(x) = F_2^{-1}(1 - F_1(x))$). Analogously, under the continuity assumption X_1 and X_2 are comonotonic if and only if $X_j = h(X_i)$ a.s. for some increasing function h. For a formal proof of these assertions, see, for example, Cuesta-Albertos et al. (1993), Proposition 2.1, or McNeil et al. (2005), p. 199f. Note that here and in the following in- and decreasing does not mean strictly increasing resp. decreasing, that is, functions like h above may have flat sections, but must not be constant. In a sense, the above result shows that co- and countermonotonicity are the strictest possible forms of positive resp. negative dependency because—at least in the continuous case—the knowledge of just one component already uniquely determines the value of the complete random vector (X_1, X_2). The second component then is nothing but an increasing or decreasing deterministic function of the first one.

After this short excursion to dependence concepts we are now ready to provide a characterization of cost-efficient resp. most-expensive strategies. The key to the solution is the covariance formula of Hoeffding (1940) (see also Lehman (1966), Lemma 2):

$$\begin{aligned} \operatorname{Cov}(X_1, X_2) &= E[X_1 X_2] - E[X_1]E[X_2] \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} F(x, y) - F_1(x)F_2(y)\, dx\, dy, \end{aligned} \tag{2.3}$$

where F is the joint distribution of the random vector (X_1, X_2) and F_1, F_2 are the corresponding marginal distributions. If we now fix the latter (and thus also $E[X_1], E[X_2]$), then it is immediately seen from (2.3) that the covariance and hence the expectation $E[X_1 X_2]$ take the smallest possible values if the joint distribution function F is minimized uniformly over $\mathbb{R}^2$. By equation (2.1), this can be achieved by setting F equal to the lower Fréchet bound, that is, X_1 and X_2 must be countermonotonic with joint distribution function $F(x_1, x_2) = F^-(x_1, x_2)$. Similarly, the expectation $E[X_1 X_2]$ can be maximized by assuming X_1 and X_2 to be comonotonic with $F(x_1, x_2) = F^+(x_1, x_2)$.

Recall that, by Definition 2.2, the cost-efficient resp. most-expensive strategies $\underline{X}_T \sim G$ and $\overline{X}_T \sim G$ are those which minimize resp. maximize the expectation $E[e^{-rT}Z_TX_T]$ among all strategies X_T possessing the same payoff-distribution G. Since the constant discount factor e^{-rT} can be neglected in this context and the state-price density Z_T, of course, must not be altered, determining the sharp lower and upper cost bounds $c(\underline{X}_T)$ and $c(\overline{X}_T)$ essentially amounts to the problem of minimizing resp. maximizing $E[Z_TX_T]$ while keeping the marginal distributions $F_1 = G$ and $F_2 = F_{Z_T}$ of the random vector (X_T, Z_T) fixed. In view of the previous discussion, the following proposition which goes back to Bernard et al. (2012), Proposition 1, is now obvious.

Proposition 2.4 *A random payoff X_T is cost-efficient iff X_T and Z_T are countermonotonic. X_T is most-expensive iff X_T and Z_T are comonotonic.*

Remark 2.1 Note that the above line of argumentation does not require X_T and Z_T to be square integrable as is assumed in Bernard et al. (2012). Only $E[X_T]$ and $E[Z_TX_T]$ must be finite ($E[Z_T] = 1$ by the definition of Z_T).

For an arbitrary payoff $X_T \sim G$ at maturity T and a state-price density Z_T with distribution function F_{Z_T}, we thus can explicitly construct the cost-efficient strategy as follows: For any uniformly distributed random variable $U \sim U(0,1)$, we know from the above that $(G^{-1}(U), F_{Z_T}^{-1}(1-U))$ is countermonotononic. Moreover, if F_{Z_T} is continuous, then $U = 1 - F_{Z_T}(Z_T) \sim U(0,1)$ and $F_{Z_T}^{-1}(F_{Z_T}(Z_T)) = Z_T$, hence $(\underline{X}_T, Z_T) = (G^{-1}(1-F_{Z_T}(Z_T)), Z_T)$ is a countermonotonic random vector with the desired margins $F_1 = G$, $F_2 = F_{Z_T}$, and fixed state-price density Z_T. Since $(G^{-1}(U), F_{Z_T}^{-1}(U))$ is a comonotonic vector with the same marginal distributions, we analogously obtain a similar representation for the most-expensive strategy $\overline{X}_T$. The following theorem, which is due to Bernard et al. (2012), summarizes the results.

Theorem 2.5 (Bernard et al. 2012, Proposition 3) *Suppose that the state-price density Z_T has a continuous distribution function F_{Z_T}. Then $\underline{X}_T = G^{-1}(1-F_{Z_T}(Z_T))$ is the cost-efficient strategy and $\overline{X}_T = G^{-1}(F_{Z_T}(Z_T))$ is the most-expensive way to achieve a payoff with given distribution function G. Moreover, for any payoff $X_T \sim G$, the lower and upper cost bounds are given by*

$$c(X_T) \geq E\left[e^{-rT}Z_T\underline{X}_T\right] = e^{-rT}\int_0^1 F_{Z_T}^{-1}(y)\,G^{-1}(1-y)\,dy,$$

$$c(X_T) \leq E\left[e^{-rT}Z_T\overline{X}_T\right] = e^{-rT}\int_0^1 F_{Z_T}^{-1}(y)\,G^{-1}(y)\,dy.$$

Remark 2.2 In the above construction we assumed the distribution F_{Z_T} of the state-price density to be continuous. However, we want to stress here that there is a way to obtain co(unter)monotonic pairs of random variables which attain the upper resp. lower Fréchet bounds and thus maximize resp. minimize the cost of a strategy even if the corresponding marginal distributions are both not continuous. This can be achieved with help of the so-called *distributional transform* which for an arbitrary random variable $Y \sim F_Y$ and some uniformly distributed $U \sim U(0,1)$ is defined by $U_Y = F_Y(Y-) + U\big(F_Y(Y) - F_Y(Y-)\big)$. Using this definition, one can show that $U_Y \sim U(0,1)$ and $Y = F_Y^{-1}(U_Y)$ a.s. (see Rüschendorf (2009), Proposition 2.1). Hence, for a random variable Y with discontinuous distribution function F_Y the distributional transform U_Y behaves in exactly the same way as $U = F_2(X_2)$ does in case of a random variable X_2 with continuous distribution function F_2. Thus, we can always bridge the gap between continuous and discontinuous distributions by replacing $F_Y(Y)$ by U_Y whenever necessary. Consequently, it is not necessary to require that the distribution function F_{Z_T} of the state-price density is continuous; the cost-efficient and most expensive strategies can more generally be defined by $\underline{X}_T = G^{-1}(1 - U_{Z_T})$ resp. $\overline{X}_T = G^{-1}(U_{Z_T})$.

2.2 Cost-efficiency in Lévy markets

Suppose now that the asset price process $(S_t)_{t\geq 0} = (S_0 e^{L_t})_{t\geq 0}$ is driven by a Lévy process $(L_t)_{t\geq 0}$. Apart from the cases where $(L_t)_{t\geq 0}$ either is a Brownian motion or a Poisson process, such a Lévy market setting is incomplete. This means that the set of possible risk-neutral martingale measures is not a singleton, but typically has uncountably many elements (see Eberlein and Jacod (1997)). Thus, one has to rely on additional optimality criteria, preference assumptions, or calibration results to real data from options markets to choose a specific martingale measure for pricing.

Throughout the paper we will use the Esscher martingale measure for this purpose which was first introduced to option pricing by Gerber and Shiu (1994). Apart from the fact that the Esscher transform provides a transparent, unambigious, and numerically very tractable way to obtain a risk-neutral measure, this choice can also be motivated from a theoretical point of view. In Keller (1997), chapter 1.4.3, it is shown that the Esscher approach can be obtained in a natural way from the assumption of the existence of a competitive equilibrium with respect to a power utility function. Moreover, there is a close relationship between Esscher transforms and minimal entropy: Esche and Schweizer (2005) prove that the Esscher martingale measure of the exponential transform of $(L_t)_{t\geq 0}$ coincides with the minimal entropy martingale measure, that is, this Esscher measure is the uniquely determined measure that minimizes the relative entropy with respect to the real-world measure P among all possible risk-neutral measures Q. An extension of this result and a precise discussion on Esscher transforms of exponential Lévy models can be found in Hubalek and Sgarra (2006). Another useful feature of Esscher transforms is the preservation of the Lévy property: $(L_t)_{t\geq 0}$ remains a Lévy process under any Esscher measure Q^θ to be defined below. An elementary proof of this important fact can be found in Raible (2000), Proposition 1.8. Chapter 1.6 of the latter thesis also provides some evidence that the Esscher martingale measure is the only one for which the corresponding state-price density $(Z_t)_{t\geq 0}$ solely depends on the current stock price S_t, but not on further previous values S_s with $0 \leq s < t$. This is of particular importance when studying cost-efficiency because it facilitates the representation and calculation of the efficient strategies. Constructing and computing the efficient payoffs $\underline{X}_T$ and $\overline{X}_T$ along the lines discussed before will become fairly difficult and cumbersome if the state-price density and hence F_{Z_T} depend on more than one random variable.

To properly define the Esscher martingale measure, the following basic assumption on the driving Lévy process $(L_t)_{t\geq 0}$ is made for the remainder of this paper.

Assumption ($\mathbb{M}$) The random variable L_1 is nondegenerate and possesses a moment generating function $M_{L_1}(u) = E[e^{uL_1}]$ on some open interval (a, b) with $a < 0 < b$ and $b - a > 1$.

This condition will turn out to be necessary (but not always sufficient) for the existence of the risk-neutral Esscher measure, as we shall see below.

Definition 2.6 Let $(L_t)_{t\geq 0}$ be a Lévy process on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. We call an *Esscher transform* any change of P to a locally equivalent measure Q^θ with a density process $Z_t^\theta = \frac{dQ^\theta}{dP}|_{\mathcal{F}_t}$ of the form

$$Z_t^\theta = \frac{e^{\theta L_t}}{M_{L_t}(\theta)}, \tag{2.4}$$

where M_{L_t} is the moment generating function of L_t as before, and $\theta \in (a, b)$.

To emphasize the dependence of the Esscher measure Q^θ and its density process $(Z_t^\theta)_{t\geq 0}$ on the parameter θ, we always add the latter as superscript. Similarly, we indicate by $E_\theta[\cdot]$ that the expectation is calculated with respect to Q^θ. Using the stationarity and independence of the increments of every Lévy process $(L_t)_{t\geq 0}$, which also imply the relation

$$M_{L_t}(u) = M_{L_1}(u)^t \quad \text{for all } u \in \mathbb{R} \text{ and } t \geq 0,$$

it is not hard to show that $(Z_t^\theta)_{t\geq 0}$ indeed is a density process for all $\theta \in (a, b)$ and, as already mentioned above, $(L_t)_{t\geq 0}$ also is a Lévy process under Q^θ for all these θ. However, the discounted stock price process $(e^{-rt}S_t)_{t\geq 0}$ will not be a martingale under all Q^θ. The parameter $\bar{\theta}$ of the *risk-neutral Esscher martingale measure* $Q^{\bar{\theta}}$ for which this property holds has to fulfill a certain condition: Since $(L_t)_{t\geq 0}$ has stationary and independent increments, the discounted price process is a martingale under $Q^{\bar{\theta}}$ if and only if the equation $S_0 = E_{\bar{\theta}}[e^{-rt}S_t]$ holds. This implies

$$\begin{aligned} S_0 &= E_{\bar{\theta}}[e^{-rt}S_t] = e^{-rt}S_0 E\big[Z_t^{\bar{\theta}} e^{L_t}\big] = e^{-rt}S_0 E\left[\frac{e^{\bar{\theta}L_t}}{M_{L_t}(\bar{\theta})}\, e^{L_t}\right] \\ &= e^{-rt}S_0 \left(\frac{M_{L_1}(\bar{\theta}+1)}{M_{L_1}(\bar{\theta})}\right)^t \end{aligned}$$

which means that $\bar{\theta}$ has to solve the equation

$$e^r = \frac{M_{L_1}(\bar{\theta}+1)}{M_{L_1}(\bar{\theta})}. \tag{2.5}$$

This also explains why it is necessary to require M_{L_1} to be defined on an interval with length greater than one. But, as mentioned before, Assumption $(\mathbb{M})$ alone does not guarantee the existence of a solution $\bar{\theta}$. The next lemma, taken from Raible (2000), Proposition 2.8, provides a sufficient condition for this and further shows that the solution, if existent, is unique.

Lemma 2.7 *If Assumption* $(\mathbb{M})$ *is in force, we have:*

a) For each $c > 0$, there is at most one $\theta \in \mathbb{R}$ such that

$$\frac{M_{L_1}(\theta+1)}{M_{L_1}(\theta)} = c.$$

b) If $\lim_{u\downarrow a} M_{L_1}(u) = \lim_{u\uparrow b} M_{L_1}(u) = \infty$, then the previous equation has exactly one solution $\theta \in (a, b-1)$ for each $c > 0$.

Remark 2.3 Note that within this framework one cannot assume the state-price density to be square integrable in general. The interval (a, b) on which the moment generating function M_{L_1} is defined and finite can be fairly small, so it might happen that a solution $\bar{\theta} \in (a, b-1)$ of (2.5) exists, but $2\bar{\theta} \notin (a, b)$, implying that $E[e^{2\bar{\theta}L_t}]$ is infinite and hence $Z_t^{\bar{\theta}}$ is not square integrable for any $t > 0$.

Observe that a negative solution $\bar{\theta} < 0$ of (2.5) corresponds to a bullish market scenario, whereas in bearish markets we have $\bar{\theta} > 0$. Heuristically, this can be seen as follows: Looking at the equation

$$S_0 e^{rt} = E_{\bar{\theta}}[S_t] = \frac{S_0}{M_{L_t}(\bar{\theta})}\, E\big[e^{(\bar{\theta}+1)L_t}\big]$$

and ignoring the denominator $M_{L_t}(\bar{\theta})$ on the right hand side for a moment, it becomes obvious that in a bullish market situation where the expected return $E[S_t/S_0] = E[e^{L_t}]$ is greater than e^{rt}, the Esscher parameter $\bar{\theta}$ must be negative to shrink the rate of return down to the risk-free rate. Similarly, in bearish markets where $E[S_t/S_0] < e^{rt}$ we must have $\bar{\theta} > 0$ to adjust the rate of return accordingly.

After these preliminaries, we can reformulate Theorem 2.5 in terms of the driving Lévy process instead of the state-price process. Observe that here and in the following we assume the present time t to equal zero. If $t > 0$, then in all formulas T has to be replaced by $T - t$.

Proposition 2.8 (Cost-efficient payoffs in Lévy models) *Let $(L_t)_{t\geq 0}$ be a Lévy process with continuous distribution function F_{L_T} at maturity $T > 0$, and assume that a solution $\bar{\theta}$ of (2.5) exists.*
If $\bar{\theta} < 0$, the cost-efficient payoff $\underline{X}_T$ and the most-expensive payoff $\overline{X}_T$ with distribution function G are given by

$$\underline{X}_T = G^{-1}(F_{L_T}(L_T)) \quad \text{and} \quad \overline{X}_T = G^{-1}(1 - F_{L_T}(L_T)). \tag{2.6}$$

If $\bar{\theta} > 0$, the cost-efficient and the most-expensive payoffs are given by

$$\underline{X}_T = G^{-1}(1 - F_{L_T}(L_T)) \quad \text{and} \quad \overline{X}_T = G^{-1}(F_{L_T}(L_T)). \tag{2.7}$$

$\underline{X}_T$ and $\overline{X}_T$ are almost surely unique.

Proof: Let $X_T \sim G$ be a payoff with distribution G, denote the distribution function of $Z_T^{\bar{\theta}} = \frac{e^{\bar{\theta} L_T}}{M_{L_T}(\bar{\theta})}$ by $F_{Z_T^{\bar{\theta}}}$ and observe that the representation of $Z_T^{\bar{\theta}}$ implies that the continuity of F_{L_T} transfers to $F_{Z_T^{\bar{\theta}}}$. From Theorem 2.5 and the discussion preceding it we already know that the cost-efficient payoff in general is given by $\underline{X}_T = G^{-1}(1 - F_{Z_T^{\bar{\theta}}}(Z_T^{\bar{\theta}}))$. If now $\bar{\theta} < 0$, then

$$\begin{aligned} 1 - F_{Z_T^{\bar{\theta}}}(x) &= 1 - P\big(Z_T^{\bar{\theta}} \leq x\big) = 1 - P\big(\bar{\theta} L_T \leq \ln\big(x M_{L_T}(\bar{\theta})\big)\big) \\ &= F_{L_T}\left(\frac{1}{\bar{\theta}} \ln\big(x M_{L_T}(\bar{\theta})\big)\right), \end{aligned}$$

from which $\underline{X}_T = G^{-1}(F_{L_T}(L_T))$ follows immediately. In a similar way one obtains $\overline{X}_T = G^{-1}(1 - F_{L_T}(L_T))$, and the representations for the case $\bar{\theta} > 0$ can be proven analogously.

To show the almost sure uniqueness of the strategies, suppose that again $\bar{\theta} < 0$ and $\underline{X}'_T \sim G$ is another cost-efficient strategy with payoff-distribution G. Then $(\underline{X}'_T, Z_T^{\bar{\theta}})$ must also be countermonotonic by Proposition 2.4, hence $\underline{X}'_T = h(Z_T^{\bar{\theta}})$ almost surely for some decreasing function h (see p. 7). On the other hand, we know that $\underline{X}_T = G^{-1}(1 - F_{Z_T^{\bar{\theta}}}(Z_T^{\bar{\theta}}))$. But since both $\underline{X}_T$ and $\underline{X}'_T$ have the same distribution function G, it must hold that $h(z) = G^{-1}(1 - F_{Z_T^{\bar{\theta}}}(z))$ for almost all $z \in \mathbb{R}$ and hence $\underline{X}_T = \underline{X}'_T$ almost surely. The proofs of uniqueness in the other cases are almost identical and therefore omitted here. $\square$

The Lévy market setting does not only allow for simpler representations of the efficient strategies, but also for more explicit and computable cost bounds as is shown in the next proposition.

Proposition 2.9 (Cost bounds in Lévy models) *Let $(L_t)_{t\geq 0}$ be a Lévy process with continuous distribution F_{L_T} at maturity $T>0$, and assume that a solution $\bar{\theta}$ of (2.5) exists.*
If $\bar{\theta}<0$, one has the following bounds for the costs of any strategy with terminal payoff $X_T \sim G$:

$$E\big[e^{-rT}Z_T^{\bar{\theta}}X_T\big] \geq E\big[e^{-rT}Z_T^{\bar{\theta}}\underline{X}_T\big] = \frac{1}{M_{L_T}(\bar{\theta})}\int_0^1 e^{\bar{\theta}F_{L_T}^{-1}(1-y)-rT}G^{-1}(1-y)\,dy,$$

$$E\big[e^{-rT}Z_T^{\bar{\theta}}X_T\big] \leq E\big[e^{-rT}Z_T^{\bar{\theta}}\overline{X}_T\big] = \frac{1}{M_{L_T}(\bar{\theta})}\int_0^1 e^{\bar{\theta}F_{L_T}^{-1}(1-y)-rT}G^{-1}(y)\,dy.$$

If $\bar{\theta}>0$, the same formulas hold true with $F_{L_T}^{-1}(1-y)$ replaced by $F_{L_T}^{-1}(y)$.

Proof: We here prove the lower bound for the case $\bar{\theta}<0$ only since the other cases are very similar. Thus, let $\bar{\theta}<0$ and observe that we then have

$$F_{Z_T^{\bar{\theta}}}(x) = 1 - F_{L_T}\left(\frac{1}{\bar{\theta}}\ln\big(xM_{L_T}(\bar{\theta})\big)\right)$$

as seen in the proof of Propostion 2.8. Hence, the inverse is given by

$$F_{Z_T^{\bar{\theta}}}^{-1}(x) = \frac{e^{\bar{\theta}F_{L_T}^{-1}(1-x)}}{M_{L_T}(\bar{\theta})}.$$

Moreover, the lower bound of the cost of a strategy with terminal payoff $X_T \sim G$ is $E\big[e^{-rT}Z_T^{\bar{\theta}}\underline{X}_T\big]$. Due to Theorem 2.5 we obtain

$$\begin{aligned} E\big[e^{-rT}Z_T^{\bar{\theta}}\underline{X}_T\big] &= \int_0^1 e^{-rT}F_{Z_T^{\bar{\theta}}}^{-1}(y)\,G^{-1}(1-y)\,dy \\ &= \frac{1}{M_{L_T}(\bar{\theta})}\int_0^1 e^{\bar{\theta}F_{L_T}^{-1}(1-y)-rT}G^{-1}(1-y)\,dy \end{aligned}$$

which proves the assertion. $\square$

On p. 11 we already saw that the sign of the risk-neutral Esscher parameter $\bar{\theta}$ characterizes the market behaviour (bullish or bearish). The size of $|\bar{\theta}|$ reflects the magnitude of the drift of the price process and thus can be regarded as a measure for the strength of the market trend. An obvious question then is to ask what impact $\bar{\theta}$ has on the cost bounds derived above, or, more precisely, how is the efficiency loss $c(X_T) - c(\underline{X}_T)$ influenced by $\bar{\theta}$? The next theorem shows that the greater the absolute value $|\bar{\theta}|$ of the risk-neutral Esscher parameter, the greater are the efficiency losses. In other words, the more pronounced the market trend is, the more profitable becomes an investment in a cost-efficient strategy. Analogously to the previous notation $E_{\bar{\theta}}[\cdot]$, we denote by $\mathrm{Cov}_{\bar{\theta}}(\cdot,\cdot)$ the covariance calculated with respect to this risk-neutral Esscher measure $Q^{\bar{\theta}}$.

Theorem 2.10 (Influence of $\bar{\theta}$ on the efficiency loss) *Let $(L_t)_{t\geq 0}$ be a Lévy process with continuous distribution F_{L_T} at maturity $T > 0$, and assume that a solution $\bar{\theta}$ of (2.5) exists. Suppose that $\underline{X}_T$ is the cost-efficient strategy associated to the payoff X_T with distribution function G. If $E_{\bar{\theta}}\big[(X_T - \underline{X}_T)^2\big] < \infty$, then the efficiency loss*

$$l(\bar{\theta}) = c(X_T) - c(\underline{X}_T) = e^{-rT} E_{\bar{\theta}}[X_T - \underline{X}_T]$$

is increasing in $|\bar{\theta}|$, that is, $l(\bar{\theta})$ is decreasing on $\mathbb{R}_-$ and increasing on $\mathbb{R}_+$.

Proof: Since $Q^{\bar{\theta}}$ is a risk-neutral measure, we have $E_{\bar{\theta}}\big[e^{L_T}\big] = e^{rT} < \infty$, and the construction of $Q^{\bar{\theta}}$ and Assumption ($\mathbb{M}$) imply $\bar{\theta} \in (a, b-1)$ with $a < 0 < b$, hence there exists a sufficiently small $\epsilon > 0$ such that $\bar{\theta} - \epsilon \in (a, b-1)$, too. Thus, we have

$$E_{\bar{\theta}}[e^{-\epsilon L_T}] = \frac{E[e^{(\bar{\theta}-\epsilon)L_T}]}{M_{L_T}(\bar{\theta})} = \frac{M_{L_T}(\bar{\theta}-\epsilon)}{M_{L_T}(\bar{\theta})} < \infty$$

and conclude that L_T also has a moment generating function $M^{\bar{\theta}}_{L_T}(u)$ under the risk-neutral Esscher measure $Q^{\bar{\theta}}$ which is well-defined and finite at least on the open interval $(-\epsilon, 1)$. In particular, this implies $E_{\bar{\theta}}[L_T^2] < \infty$ and hence $E_{\bar{\theta}}[|L_T|] < \infty$.

Thus, we can differentiate the state-price density $Z^{\bar{\theta}}_T = \frac{e^{\bar{\theta}L_T}}{M_{L_T}(\bar{\theta})}$ with respect to $\bar{\theta}$ and obtain

$$\begin{aligned} \frac{\partial Z^{\bar{\theta}}_T}{\partial \bar{\theta}} &= \frac{L_T e^{\bar{\theta}L_T} M_{L_T}(\bar{\theta}) - e^{\bar{\theta}L_T} M'_{L_T}(\bar{\theta})}{M_{L_T}(\bar{\theta})^2} \\ &= Z^{\bar{\theta}}_T L_T - \frac{e^{\bar{\theta}L_T} E[L_T e^{\bar{\theta}L_T}]}{M_{L_T}(\bar{\theta})^2} = Z^{\bar{\theta}}_T (L_T - E_{\bar{\theta}}[L_T]) \end{aligned}$$

where in the second equality we used the fact that $M'_{L_T}(\bar{\theta}) = E\big[\frac{\partial}{\partial\bar{\theta}} e^{\bar{\theta}L_T}\big]$. The interchange between differentiation and integration here is justified because $E\big[\big|\frac{\partial}{\partial\bar{\theta}} e^{\bar{\theta}L_T}\big|\big] = E[|L_T| e^{\bar{\theta}L_T}] = M_{L_T}(\bar{\theta}) E_{\bar{\theta}}[|L_T|] < \infty$ as shown above.

Further, observe that X_T does not depend on $\bar{\theta}$, and by Proposition 2.8, neither does $\underline{X}_T$, such that we have $\frac{\partial Z_T^{\bar{\theta}}(X_T-\underline{X}_T)}{\partial\bar{\theta}} = \frac{\partial Z_T^{\bar{\theta}}}{\partial\bar{\theta}}(X_T-\underline{X}_T)$ and

$$\begin{aligned}
E\left[\left|\frac{\partial Z_T^{\bar{\theta}}}{\partial\bar{\theta}}(X_T-\underline{X}_T)\right|\right] &= E\big[\big|Z_T^{\bar{\theta}}(L_T - E_{\bar{\theta}}[L_T])(X_T-\underline{X}_T)\big|\big] \\
&\leq E\big[Z_T^{\bar{\theta}}(|L_T| + E_{\bar{\theta}}[|L_T|])\,|X_T-\underline{X}_T|\big] \\
&= E\big[Z_T^{\bar{\theta}}|L_T|\,|X_T-\underline{X}_T|\big] + E_{\bar{\theta}}\big[|L_T|\big]E\big[Z_T^{\bar{\theta}}\,|X_T-\underline{X}_T|\big] \\
&= E_{\bar{\theta}}\big[|L_T|\,|X_T-\underline{X}_T|\big] + E_{\bar{\theta}}\big[|L_T|\big]E_{\bar{\theta}}\big[|X_T-\underline{X}_T|\big] \\
&< \infty
\end{aligned}$$

because $E_{\bar{\theta}}[L_T^2] < \infty$ and, by assumption, also $E_{\bar{\theta}}\big[(X_T-\underline{X}_T)^2\big] < \infty$. This again allows to interchange differentiation and integration in the following calculation which yields, similarly as above,

$$\begin{aligned}
\frac{\partial l(\bar{\theta})}{\partial\bar{\theta}} &= e^{-rT}\frac{\partial E\big[Z_T^{\bar{\theta}}(X_T-\underline{X}_T)\big]}{\partial\bar{\theta}} = e^{-rT}E\left[\frac{\partial Z_T^{\bar{\theta}}}{\partial\bar{\theta}}\,(X_T-\underline{X}_T)\right] \\
&= e^{-rT}E\big[Z_T^{\bar{\theta}}(L_T - E_{\bar{\theta}}[L_T])(X_T-\underline{X}_T)\big] \\
&= e^{-rT}\Big(E\big[Z_T^{\bar{\theta}}L_T(X_T-\underline{X}_T)\big] - E_{\bar{\theta}}\big[L_T\big]E\big[Z_T^{\bar{\theta}}(X_T-\underline{X}_T)\big]\Big) \\
&= e^{-rT}\mathrm{Cov}_{\bar{\theta}}(L_T, X_T-\underline{X}_T).
\end{aligned}$$

Hence, $l(\bar{\theta})$ is increasing in $\bar{\theta}$ iff $\mathrm{Cov}_{\bar{\theta}}(L_T, X_T) \geq \mathrm{Cov}_{\bar{\theta}}(L_T, \underline{X}_T)$. The latter inequality is fulfilled for $\bar{\theta} > 0$, because by Proposition 2.8 we then have $\underline{X}_T = G^{-1}(1-F_{L_T}(L_T))$ almost surely, so $\underline{X}_T$ is a decreasing function of L_T, implying that $(L_T, \underline{X}_T)$ is a countermonotonic pair of random variables, and thus, has the smallest covariance among all pairs of random variables possessing the same marginal distributions (see p. 7). Analogously one obtains that $l(\bar{\theta})$ is decreasing for $\bar{\theta} < 0$, because in this case $\underline{X}_T = G^{-1}(F_{L_T}(L_T))$ is an increasing function of L_T by Proposition 2.8. Therefore, $(L_T, \underline{X}_T)$ is comonotonic and hence $\mathrm{Cov}_{\bar{\theta}}(L_T, X_T) \leq \mathrm{Cov}_{\bar{\theta}}(L_T, \underline{X}_T)$. □

The ideas in the last part of the above proof also allow to obtain another simple but very useful characterization of cost-efficiency in Lévy models.

Corollary 2.11 (Characterization of cost-efficiency in Lévy models) *Let $(L_t)_{t\geq 0}$ be a Lévy process with continuous distribution F_{L_T} at maturity $T > 0$, and assume that a solution $\bar{\theta}$ of (2.5) exists.*

i) If $\bar{\theta} < 0$, a payoff $X_T \sim G$ is cost-efficient if and only if it is increasing in L_T.

ii) If $\bar{\theta} > 0$, a payoff $X_T \sim G$ is cost-efficient if and only if it is decreasing in L_T.

For the most-expensive strategy, the reverse holds true.

Proof: Suppose that $\bar{\theta} < 0$ and $\underline{X}_T$ is a cost-efficient payoff with distribution function G. On the one hand, due to Proposition 2.8, it holds that $\underline{X}_T = G^{-1}(F_{L_T}(L_T))$ almost surely, such that $\underline{X}_T$ is increasing in L_T, i.e., $X_T = v(L_T)$ for some measurable, increasing function v (recall that we do not understand in- and decreasing in the strict sense here, see also p. 7). On the other hand, if a payoff X_T is increasing in L_T, then the representation $Z_T^{\bar{\theta}} = \frac{e^{\bar{\theta} L_T}}{M_{L_T}(\bar{\theta})}$, together with the assumption $\bar{\theta} < 0$, implies that $X_T = h(Z_T)$ where $h(z) = v\big(\bar{\theta}^{-1} \ln\big(z\, M_{L_T}(\bar{\theta})\big)\big)$ is decreasing. Therefore, X_T and Z_T are countermonotonic and hence X_T is cost-efficient due to Proposition 2.4. The second statement as well as the reverse for the most-expensive strategies can be shown analogously. □

Example 2.12

i) Applying Corollary 2.11 to the special payoff $X_T = S_T = S_0 e^{L_T}$ one obtains that buying one stock for S_0 at time $t = 0$ is a cost-efficient way to achieve a payoff with distribution $G = F_{S_T}$ at time T if and only if $\bar{\theta} < 0$.

ii) Assume again the setting of Corollary 2.11 and consider the payoff of a European put option at maturity time $T > 0$ with strike $K > 0$, i.e., $X_T^P = (K - S_T)_+ = (K - S_0 e^{L_T})_+$. Clearly, X_T^P is a decreasing function of L_T and hence is cost-efficient if and only if $\bar{\theta} > 0$. For $\bar{\theta} < 0$, however, the classical put is the most-expensive way to realize a payoff with distribution $G = F_{X_T^P}$. Similarly, the payoff $X_T^C = (S_0 e^{L_T} - K)_+$ of a European call option with strike K and maturity T is cost-efficient if $\bar{\theta} < 0$ and most-expensive if $\bar{\theta} > 0$.

The latter corollary also implies the inefficieny of path-dependent payoffs. Here we call a payoff X_T *path-dependent* if X_T does not solely depend on the asset price S_T at maturity time T (or equivalently on L_T), but at least on one more values S_t, resp. L_t, with $0 < t < T$. Consequently, a path-dependent payoff never is an increasing or decreasing function of L_T alone, and therefore, cannot be cost-efficient either in general. The only exception is the case $\bar{\theta} = 0$ which implies $Z_t^{\bar{\theta}} \equiv 1$ for all $0 \le t \le T$, thus $P = Q^0$ already is a risk-neutral measure itself. As is immediately obvious from Definition 2.2, the possible price range $[c(\underline{X}_T), c(\overline{X}_t)]$ of any payoff $X_T \sim G$ then shrinks to a singleton or, in other words, for $\bar{\theta} = 0$ every payoff X_T already is cost-efficient and cannot be improved further. Summing up, we obtain the following generalization of Bernard et al. (2012), Corollary 3, to the Lévy market setting.

Corollary 2.13 (Inefficiency of path-dependent payoffs) *Suppose that $(L_t)_{t \ge 0}$ is a Lévy process with continuous distribution F_{L_T} at maturity $T > 0$ and that a solution $\bar{\theta}$ of* (2.5) *exists, then path-dependent payoffs are not cost-efficient unless $\bar{\theta} = 0$.*

Remark 2.4 In some settings, path-dependent payoffs X_T can be improved by conditioning on S_T resp. L_T. Vanduffel et al. (2009) proved that risk-averse investors with fixed investment horizon will always prefer the payoff $X'_T = E[X_T \mid S_T]$ to X_T in a Lévy market model where the real-world and risk-neutral measures P and Q are related by an Esscher transformation. More generally, path-dependent payoffs are suboptimal for risk-averse investors in any setting where the state-price density is a function of S_T, see Kassberger and Liebmann (2011). Observe that the improved payoff X'_T is no longer path-dependent due to the conditioning on S_T, hence it fits into the present framework and may be enhanced further by applying Proposition 2.8. In Vanduffel et al. (2012), this approach is applied to Dollar cost averaging which is shown to be outperformed by a static strategy of investing in a suitable portfolio of path-independent options. Some general comparison results for prices of path-dependent options like Asian or lookback options are given in Bergenthum and Rüschendorf (2008).

2.3 Models for the Lévy process

In the last two decades more and more researchers started to use jump-diffusions and, more generally, Lévy processes as a valuable and flexible tool to model asset price processes as well as the term structure of interest rates. These typically provide a much better fit to real market data because the inherent jumps allow for a more realistic modeling and quantification of the risk of large price movements within short time intervals which are often severly underestimated in a pure diffusion framework. A comprehensive overview on the most prominent Lévy processes that already have been applied to financial modeling can be found in the books of Schoutens (2003) and Cont and Tankov (2004), for jump-diffusion models we also refer to Kou (2002).

From the wide range of potential models mentioned there, we concentrate on the normal inverse Gaussian (NIG) and Variance Gamma (VG) Lévy processes in the following. Both are subclasses, resp. limiting cases, of the greater class of generalized hyperbolic processes which were introduced to finance in Eberlein (2001) and Eberlein and Prause (2002). Moreover, we include the Brownian motion as a benchmark model here which allows us to compare the prices of cost-efficient strategies within the NIG and VG models with those that can be achieved in the classical Black–Scholes framework. Next we recall some basic facts and properties of these processes which will be used in subsequent sections.

Normal inverse Gaussian model. The normal inverse Gaussian process was first applied to finance in Barndorff-Nielsen (1995) and Barndorff-Nielsen (1998). Its generating distributions form a subclass of the family of generalized hyperbolic distributions $GH(\lambda, \alpha, \beta, \delta, \mu)$ which one gets by choosing $\lambda = -0.5$ and that can be obtained as a normal mean-variance mixture with an inverse Gaussian mixing distribution. More specifically, if $X \sim NIG(\alpha, \beta, \delta, \mu)$, then the random variable X can be represented as follows:

$$X \stackrel{d}{=} \mu + \beta Z + \sqrt{Z}\, W, \tag{2.8}$$

where $\mu \in \mathbb{R}$, $W \sim N(0,1)$, and $Z \sim IG\big(\delta, \sqrt{\alpha^2 - \beta^2}\big)$ is an inverse Gaussian distributed random variable with $\delta > 0$ and $0 \leq |\beta| < \alpha$ that is independent of W. This representation also entails that the infinite divisibility of the mixing inverse Gaussian distribution transfers to the NIG mixture distribution, thus there exists a Lévy process $(L_t)_{t\geq 0}$ with $\mathcal{L}(L_1) = NIG(\alpha, \beta, \delta, \mu)$. The Lebesgue density $d_{NIG(\alpha,\beta,\delta,\mu)}$ can be obtained by calculating

$$\begin{aligned} d_{NIG(\alpha,\beta,\delta,\mu)}(x) &= \int_0^\infty d_{N(\mu+\beta y)}(x)\, d_{IG(\delta,\sqrt{\alpha^2-\beta^2})}(y)\, dy \\ &= n(\alpha,\beta,\delta)\, \frac{K_1\big(\alpha\sqrt{\delta^2 + (x-\mu)^2}\big)}{\sqrt{\delta^2 + (x-\mu)^2}}\, e^{\beta(x-\mu)}, \end{aligned}$$

where $K_1(x)$ is the modified Bessel function of third kind with index 1, and the norming constant $n(\alpha, \beta, \delta)$ is given by

$$n(\alpha,\beta,\delta) = \frac{\alpha\delta}{\pi}\, e^{\delta\sqrt{\alpha^2-\beta^2}}.$$

The corresponding moment generating function $M_{NIG(\alpha\beta,\delta,\mu)}$ then can easily be derived observing that

$$\begin{aligned} M_{NIG(\alpha,\beta,\delta,\mu)}(u) &= \int_{-\infty}^{\infty} e^{ux} d_{NIG(\alpha,\beta,\delta,\mu)}(x)\, dx \\ &= \int_{-\infty}^{\infty} e^{u\mu} \frac{n(\alpha,\beta,\delta)}{n(\alpha,\beta+u,\delta)}\, d_{NIG(\alpha,\beta+u,\delta,\mu)}(x)\, dx \\ &= e^{u\mu} \frac{n(\alpha,\beta,\delta)}{n(\alpha,\beta+u,\delta)} = e^{u\mu+\delta\sqrt{\alpha^2-\beta^2}-\delta\sqrt{\alpha^2-(\beta+u)^2}} \end{aligned} \tag{2.9}$$

which obviously is defined for all $u \in (-\alpha-\beta, \alpha-\beta)$. Hence, Assumption ($\mathbb{M}$) is fulfilled if $\alpha - \beta - (-\alpha - \beta) = 2\alpha > 1$. However, we have

$$\lim_{u \to \pm\alpha-\beta} M_{NIG(\alpha,\beta,\delta,\mu)}(u) = e^{(\pm\alpha-\beta)\mu+\delta\sqrt{\alpha^2-\beta^2}},$$

that is, the moment generating function tends to a finite limit at the boundaries of this interval. According to Lemma 2.7, it thus may not always be possible to find a solution $\bar{\theta}$ of equation (2.5). If it exists, it is given by

$$\bar{\theta}_{NIG} = -\frac{1}{2} - \beta + \frac{r-\mu}{\delta}\sqrt{\frac{\alpha^2}{1+(\frac{r-\mu}{\delta})^2} - \frac{1}{4}}. \tag{2.10}$$

A detailed derivation of this expression can be found in Appendix A. Note that the characteristic function ϕ_{NIG} of an NIG distribution can be obtained via the relation $\phi_{NIG}(u) = M_{NIG}(iu)$. Since for every Lévy process it holds that $\phi_{L_t}(u) = \phi_{L_1}(u)^t$, one immediately obtains from (2.9) that $\phi_{L_t}(u) = \phi_{NIG(\alpha,\beta,\delta,\mu)}(u)^t = \phi_{NIG(\alpha,\beta,\delta t,\mu t)}(u)$, hence for an NIG Lévy process $(L_t)_{t\geq 0}$ we have $\mathcal{L}(L_t) = NIG(\alpha,\beta,\delta t,\mu t)$ for all $t > 0$. Similar arguments as used above in (2.9) then show that for any Esscher transform the density $d^{\theta}_{L_t}$ of L_t under the measure Q^{θ} is

$$\begin{aligned} d^{\theta}_{L_t}(x) &= \frac{e^{\theta x}}{M_{NIG(\alpha,\beta,\delta t,\mu t)}(\theta)}\, d_{NIG(\alpha,\beta,\delta t,\mu t)}(x) \\ &= \frac{n(\alpha,\beta+\theta,\delta t)}{n(\alpha,\beta,\delta t)}\, e^{\theta(x-\mu t)} d_{NIG(\alpha,\beta,\delta t,\mu t)}(x) = d_{NIG(\alpha,\beta+\theta,\delta t,\mu t)}(x), \end{aligned} \tag{2.11}$$

that is, $(L_t)_{t\geq 0}$ remains an NIG Lévy process under every Esscher transform Q^{θ}, but with different parameter $\beta \rightsquigarrow \beta + \theta$.

Variance Gamma model. The class of Variance Gamma distributions was introduced in Madan and Senata (1990) as a more realistic model for stock return distributions. In this and the following paper, Madan and Milne (1991), where an option pricing formula for this model was derived, only symmetric VG distributions were considered. The general case including skewness was studied in Madan et al. (1998). Similar to NIG distributions, a Variance Gamma distributed random variable $X \sim VG(\lambda,\alpha,\beta,\mu)$ can be represented as a normal mean-variance mixture as in equation (2.8), but in this case the mixing variable $Z \sim G\big(\lambda, \frac{\alpha^2-\beta^2}{2}\big)$ is Gamma distributed with shape parameter $\lambda > 0$ and scale parameter $\frac{\alpha^2-\beta^2}{2}$ where $0 \leq |\beta| < \alpha$. Again, the infinite divisibility of $G\big(\lambda, \frac{\alpha^2-\beta^2}{2}\big)$ transfers to $VG(\lambda,\alpha,\beta,\mu)$. Analogously as above, the corresponding Lebesgue density $d_{VG(\lambda,\alpha,\beta,\mu)}$ can be calculated by

$$\begin{aligned} d_{VG(\lambda,\alpha,\beta,\mu)}(x) &= \int_0^{\infty} d_{N(\mu+\beta y)}(x)\, d_{G(\lambda,(\alpha^2-\beta^2)/2)}(y)\, dy \\ &= m(\lambda,\alpha,\beta)\, |x-\mu|^{\lambda-\frac{1}{2}}\, K_{\lambda}(\alpha|x-\mu|)\, e^{\beta(x-\mu)} \end{aligned}$$

with the norming constant

$$m(\lambda,\alpha,\beta) = \frac{(\alpha^2-\beta^2)^{\lambda}}{\sqrt{\pi}(2\alpha)^{\lambda-\frac{1}{2}}\Gamma(\lambda)},$$

and with the same reasoning as in (2.9) one obtains the moment generating function

$$M_{VG(\lambda,\alpha,\beta,\mu)}(u) = e^{u\mu}\frac{m(\lambda,\alpha,\beta)}{m(\lambda,\alpha,\beta+u)} = e^{u\mu}\left(\frac{\alpha^2-\beta^2}{\alpha^2-(\beta+u)^2}\right)^{\lambda} \tag{2.12}$$

which again is defined for all $u \in (-\alpha-\beta, \alpha-\beta)$. Observe that here we have $\lim_{u\to\pm\alpha-\beta} M_{VG(\lambda,\alpha,\beta,\mu)}(u) = \infty$, such that due to Lemma 2.7 b) the condition $2\alpha > 1$ is sufficient to guarantee a unique solution $\bar{\theta}$ of equation (2.5) in the VG case. Some lengthy calculations which we postpone to Appendix A show that it is given by

$$\bar{\theta}_{VG} = \begin{cases} -\frac{1}{2}-\beta, & r=\mu, \\ -\frac{1}{1-e^{-\frac{r-\mu}{\lambda}}} - \beta + \operatorname{sign}(r-\mu)\sqrt{\frac{e^{-\frac{r-\mu}{\lambda}}}{\left(1-e^{-\frac{r-\mu}{\lambda}}\right)^2}+\alpha^2}, & r\neq\mu. \end{cases} \tag{2.13}$$

Again, we have $\phi_{VG}(u) = M_{VG}(iu)$ and conclude from (2.12) that $\phi_{L_t}(u) = \phi_{VG(\lambda,\alpha,\beta,\mu)}(u)^t = \phi_{VG(\lambda t,\alpha,\beta,\mu t)}(u)$, i.e., for a VG Lévy process $(L_t)_{t\geq 0}$ it holds that $\mathcal{L}(L_t) = VG(\lambda t, \alpha, \beta, \mu t)$ for all $t > 0$. Similarly as in the NIG case, one can also show that every Esscher transform of the real-world measure P only affects the parameter β and $(L_t)_{t\geq 0}$ remains a VG Lévy process under Q^{θ}, but with different parameter $\beta + \theta$.

Remark 2.5 In many papers dealing with Variance Gamma distributions, especially the ones of Madan and coauthors, a different parametrization $VG(\sigma, \nu, \theta, \tilde{\mu})$ is used (the VG parameter θ here should not be confused with the Esscher parameter). This is related to ours as follows:

$$\sigma^2 = \frac{2\lambda}{\alpha^2-\beta^2}, \quad \nu = \frac{1}{\lambda}, \quad \theta = \beta\sigma^2 = \frac{2\beta\lambda}{\alpha^2-\beta^2}, \quad \tilde{\mu} = \mu.$$

Samuelson model. The classical benchmark model which also is at the basis of the Black–Scholes theory is to assume that the stock price process $(S_0 e^{L_t})_{t\geq 0}$ follows a geometric Brownian motion. In this case, the driving Lévy process is given by

$$L_t = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t, \quad t > 0,$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion under the physical measure P, μ is the drift and σ the volatility parameter. Here we have $\mathcal{L}(L_t) = N\big((\mu - \frac{\sigma^2}{2})t, \sigma^2 t\big)$, and the moment generating function of L_1 is well-known to equal

$$M_{N(\mu-\frac{\sigma^2}{2},\sigma^2)}(u) = e^{u(\mu-\frac{\sigma^2}{2})+\frac{u^2\sigma^2}{2}}.$$

Apparently, it is defined for all $u \in \mathbb{R}$ and tends to infinity if $u \to \pm\infty$, so Lemma 2.7 b) assures that a unique solution $\bar{\theta}$ of equation (2.5) exists. The latter here becomes

$$e^r = \frac{M_{L_1}(\bar{\theta}_N + 1)}{M_{L_1}(\bar{\theta}_N)} = \frac{e^{(\mu - \frac{\sigma^2}{2})(\bar{\theta}_N + 1) + (\bar{\theta}_N + 1)^2 \frac{\sigma^2}{2}}}{e^{(\mu - \frac{\sigma^2}{2})\bar{\theta}_N + \bar{\theta}_N^2 \frac{\sigma^2}{2}}} = e^{\mu + \bar{\theta}_N \sigma^2}$$

from which it is immediately seen that $\bar{\theta}_N = \frac{r-\mu}{\sigma^2}$.

Remark 2.6 Since this model is complete and thus the risk-neutral measure Q is unique, the Esscher density process $(Z_t^{\bar{\theta}_N})_{t \geq 0}$ here must coincide with the state-price density process $(Z_t)_{t \geq 0}$ obtained from Girsanov's theorem. This indeed is the case, as the following equation shows:

$$Z_t = \frac{e^{\frac{r-\mu}{\sigma} B_t}}{e^{\frac{(r-\mu)^2}{2\sigma^2} t}} = \frac{e^{\frac{r-\mu}{\sigma} B_t}}{E[e^{\frac{r-\mu}{\sigma} B_t}]} = \frac{e^{\frac{r-\mu}{\sigma^2}(L_t - t(\mu - \frac{\sigma^2}{2}))}}{E\left[e^{\frac{r-\mu}{\sigma^2}(L_t - t(\mu - \frac{\sigma^2}{2}))}\right]} = \frac{e^{\bar{\theta}_N L_t}}{M_{L_t}(\bar{\theta}_N)} = Z_t^{\bar{\theta}_N}.$$

3 Applications

In this section we apply the theoretical results obtained so far to some common payoff-distributions. More specifically, we consider European put and call options, self-quanto calls and puts, forwards as well as bull and bear spreads. Moreover, we provide some numerical results for the Lévy market settings discussed in Section 2.3. These calculations are based on estimated parameters from German stock price data for Allianz and Volkswagen from May 28, 2010, to September 28, 2012, which are shown in Figure 1. The estimated parameters from the daily log-returns of Allianz and Volkswagen are given in Table 1 below. The interest rate used to calculate $\bar{\theta}$ is $r = 4.2027 \cdot 10^{-6}$ which corresponds to the continuously compounded 1-Month-Euribor rate of October 1, 2012.

3.1 Put options ($\bar{\theta} < 0$)

Consider a long put option with strike $K > 0$ and maturity $T > 0$ whose payoff is $X_T^P = (K - S_T)_+ = (K - S_0 e^{L_T})_+$. As already remarked in Example 2.12 ii), X_T^P is monotonically decreasing in L_T, therefore the put option is inefficient if $\bar{\theta} < 0$ due to Corollary 2.11. The distribution function $G_P = F_{X_T^P}$ of the put payoff can easily be shown to equal

$$G_P(x) = P\big(X_T^P \leq x\big) = \begin{cases} 1 & , \text{ if } x \geq K, \\ 1 - F_{L_T}\big(\ln\big(\frac{K-x}{S_0}\big)\big) & , \text{ if } 0 \leq x < K, \\ 0 & , \text{ if } x < 0. \end{cases}$$

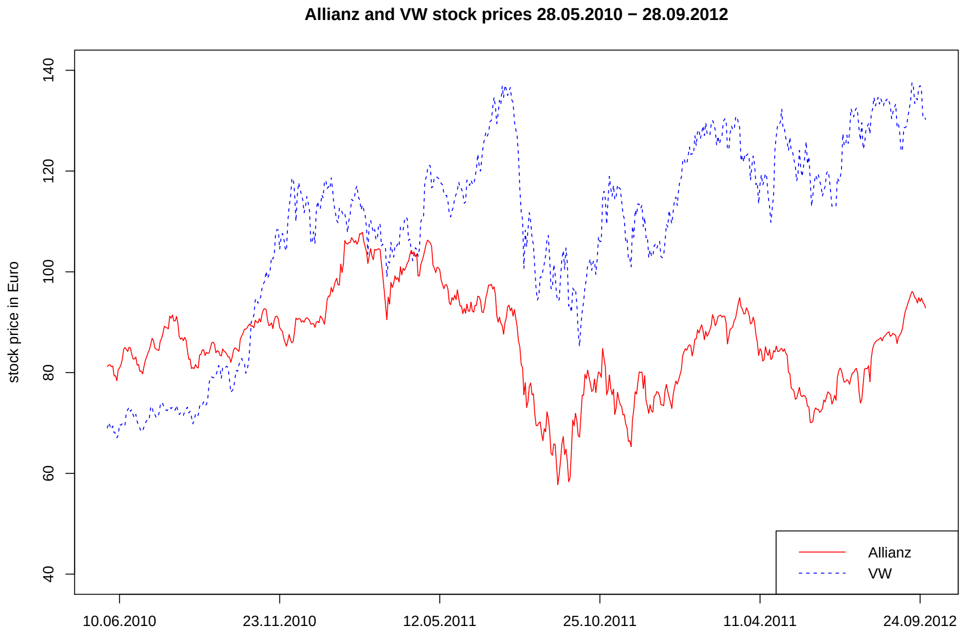

Figure 1: Daily closing prices of Allianz and Volkswagen used for parameter estimation.

Allianz	λ	α	β	δ	μ	$\bar{\theta}$
NIG	-0.5	35.020	-0.369	0.015	0.000376	-1.0127
VG	1.031	72.011	0.552	0.0	$1.941 \cdot 10^{-8}$	-1.0412
Normal	$\mu = 4.2757 \cdot 10^{-4}$, $\sigma = 0.0203$					-1.0314
Volkswagen	λ	α	β	δ	μ	$\bar{\theta}$
NIG	-0.5	48.859	-0.842	0.0231	0.001451	-2.7087
VG	1.602	82.948	-2.165	0.0	0.00206	-2.7395
Normal	$\mu = 0.00129$, $\sigma = 0.0216$					-2.7447

Table 1: Estimated parameters from daily log-returns of Allianz and Volkswagen for the NIG-, the VG-, and the Samuelson model. The interest rate used to calculate the Esscher parameter $\bar{\theta}$ in the last column is the continuously compounded 1-Month-Euribor rate of October 1, 2012, which is $r = 4.2027 \cdot 10^{-6}$.

Its inverse is given by

$$G_P^{-1}(y) = \left(K - S_0 e^{F_{L_T}^{-1}(1-y)}\right)_+, \quad y \in (0,1), \tag{3.1}$$

which follows from solving the equation

$$1 - F_{L_T}\left(\ln\left(\frac{K-x}{S_0}\right)\right) = y$$

for x and noting that x must be non-negative since the range of X_T^P is $[0, K]$. Applying Proposition 2.8, for $\bar{\theta} < 0$ the cost-efficient payoff that generates the same distribution G_P as the long put is

$$\underline{X}_T^P = G_P^{-1}(F_{L_T}(L_T)) = \left(K - S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}\right)_+. \tag{3.2}$$

Figure 2 displays the payoff X_T^P of a long put option on one Allianz stock with strike $K = 98$ and maturity $T = 23$ days, and its cost-efficient counterparts $\underline{X}_T^P$ for the three Lévy models under consideration. Although the latter payoff profiles look quite similar, a closer look reveals that the optimal payoff is model-dependent and slightly varies between the different models.

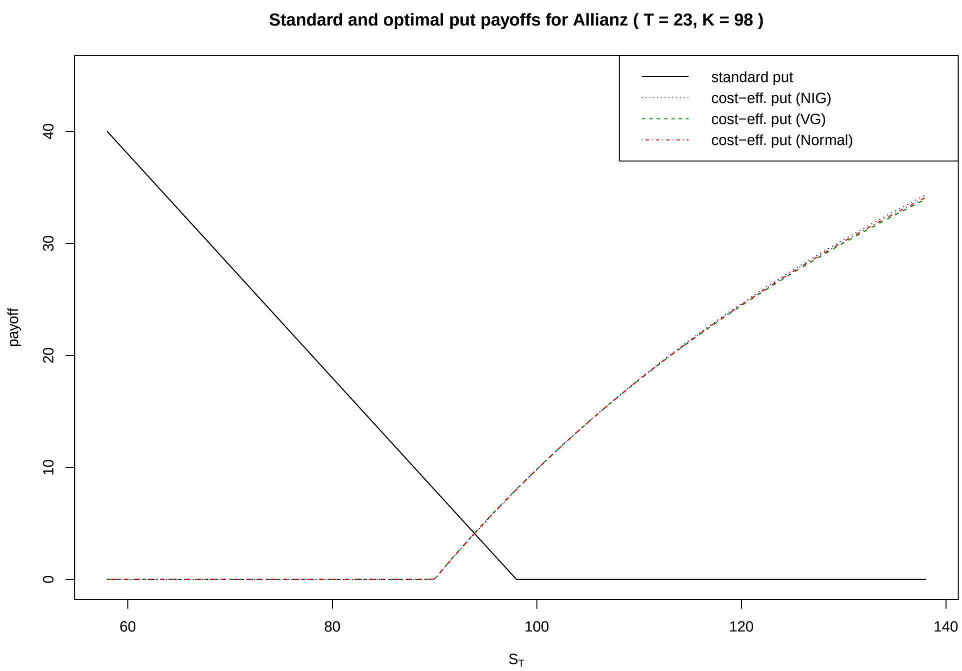

Figure 2: Payoff functions of a classical put and its cost-efficient counterparts for Allianz. The initial stock price is $S_0 = 93.42$, the closing price of Allianz at October 1, 2012. The other parameters used to calculate the cost-efficient payoffs can be found in Table 1.

Remark 3.1 Observe that the distribution function G_P and its inverse G_P^{-1} depend on the time to maturity. As already remarked on page 12, if the present time t is greater than zero, one has to replace T by $T-t$. This also implies that the efficient put payoff is not static, but a time-varying function which might be surprising at first glance. However, this is an essential and quite natural feature: Suppose a trader has bought a cost-efficient put with maturity T at time $t_0 = 0$ and wants to sell it again at some later point in time. Then a second trader who buys this efficient put from the first one at time $t_1 > t_0$ wants it, of course, to guarantee him the same payoff-distribution that a vanilla put with maturity T would provide at time t_1 which is possible if and only if the payoff of the efficient put at time t_1 is given by $\underline{X}_{T-t_1}^P = \left(K - S_{t_1} e^{F_{L_{T-t_1}}^{-1}(1-F_{L_{T-t_1}}(L_{T-t_1}))}\right)_+$, and not by $\underline{X}_T^P$ from (3.2). The variation of the cost-efficient put payoff subject to different times to maturity is illustrated in Figure 3 below.

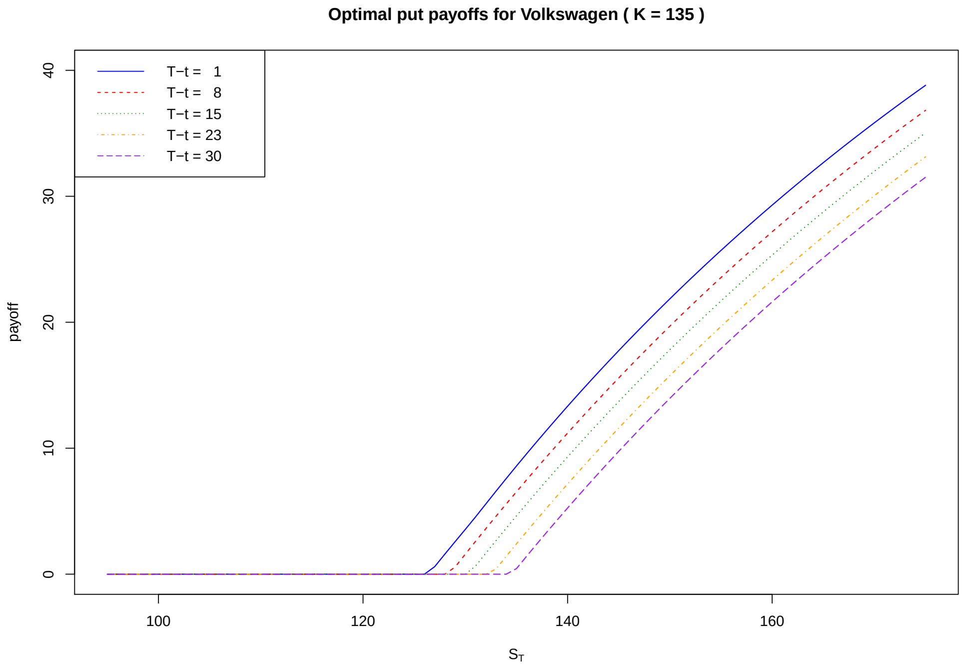

Figure 3: Cost-efficient put payoffs for different times to maturity within the VG model for Volkswagen. The initial stock price is always assumed to be $S_t = 130.55$, and the VG parameters of Volkswagen used to calculate the efficient payoffs can be found in Table 1.

Recall that Figures 2 and 3 correspond to a bullish market situation where $\bar{\theta} < 0$ (see Table 1) such that the classical put with payoff X_T^P is the most-expensive way to realize the payoff-distribution G_P. However, if the market behaviour should suddenly switch from bullish to bearish, that is, if the risk-neutral Esscher parameter $\bar{\theta}$ derived from market data should change its sign during the lifetime of the contract, then the roles of the payoffs are reversed: $X^P_{T-t_s}$ becomes cost-efficient, and the formerly efficient payoff $\underline{X}^P_{T-t_s}$ is most-expensive from that "switching time" t_s onwards. In other words, an initially optimal strategy may turn into the worst case if the market scenario significantly changes in between. This suggests that the present definition and construction of cost-efficient strategies might be extended to a more dynamic version that allows to accordingly react to reverse market movements. We do not exploit this idea further here, but leave it to future research.

Since the payoff function $X_T^{-P} = -(K - S_0 e^{L_T})_+$ of a short put with strike K and maturity T is monotonically increasing in L_T, a short put is cost-efficient if $\bar{\theta} < 0$ and most-expensive if $\bar{\theta} > 0$. Analogously to the calculations for the long put, one can show that the inverse of the distribution G_{-P} of a short put is given by

$$G_{-P}^{-1}(y) = \left(S_0 e^{F_{L_T}^{-1}(y)} - K\right)_-, \quad y \in (0,1), \tag{3.3}$$

where $(x-y)_- := -(y-x)_+$. Applying Proposition 2.8, the cost-efficient payoff that generates the same distribution G_{-P} as the short put option for $\bar{\theta} > 0$ thus is

$$\underline{X}_T^{-P} = G_{-P}^{-1}(1 - F_{L_T}(L_T)) = \left(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} - K\right)_-. \tag{3.4}$$

Intuitively, the cost-efficient strategy corresponding to a long put should coincide with a short position in the most-expensive strategy for a short put as investors simply take opposite positions. The one who is long has to pay for entering the put while the one who is short receives an upfront payment. Thus, for the long position it is optimal to minimize the costs of the put while the investor who is short wants to maximize the initial cash inflow. We formalize this observation in the following corollary.

Corollary 3.1 *Let $X_T^P = (K - S_T)_+$ be the payoff of a long put with distribution G_P and $X_T^{-P} = -(K - S_T)_+$ be the payoff of the corresponding short position with distribution G_{-P}. If the assumptions of Proposition 2.8 are in force and $\bar{\theta} < 0$, it holds that*

$$\underline{X}_T^P = -\overline{X}_T^{-P} \quad \textit{and} \quad c\big(\underline{X}_T^P\big) = -c\big(\overline{X}_T^{-P}\big).$$

Similarly, if $\bar{\theta} > 0$ we have $\underline{X}_T^{-P} = -\overline{X}_T^P$ as well as $c\big(\underline{X}_T^{-P}\big) = -c\big(\overline{X}_T^P\big)$.

Proof: Suppose that $\bar{\theta} < 0$. From equations (3.1) and (3.3) we know that

$$G_P^{-1}(y) = \left(K - S_0 e^{F_{L_T}^{-1}(1-y)}\right)_+ = -\left(S_0 e^{F_{L_T}^{-1}(1-y)} - K\right)_- = -G_{-P}^{-1}(1-y).$$

Applying Proposition 2.8 yields

$$\underline{X}_T^P = G_P^{-1}(F_{L_T}(L_T)) = -G_{-P}^{-1}(1-F_{L_T}(L_T)) = -\overline{X}_T^{-P}$$

and, due to Proposition 2.9,

$$\begin{aligned} c\big(\underline{X}_T^P\big) = E\big[e^{-rT}Z_T\underline{X}_T^P\big] &= \frac{1}{M_{L_T}(\bar\theta)}\int_0^1 e^{\bar\theta F_{L_T}^{-1}(1-y)-rT}G_P^{-1}(1-y)\,dy \\ &= -\frac{1}{M_{L_T}(\bar\theta)}\int_0^1 e^{\bar\theta F_{L_T}^{-1}(1-y)-rT}G_{-P}^{-1}(y)\,dy \\ &= -E\big[e^{-rT}Z_T\overline{X}_T^{-P}\big] = -c\big(\overline{X}_T^{-P}\big). \end{aligned}$$

The assertions for the case $\bar\theta > 0$ can be shown completely analogously. □

In Table 2 below we compare the costs of a long put option on Allianz and Volkswagen with their cost-efficient counterparts for the Lévy models discussed in Section 2.3. All computations are based on the estimated parameters given in Table 1 above. The initial stock prices S_0 of Allianz resp. Volkswagen are the closing prices at October 1, 2012, and the time to maturity is chosen to be $T = 23$ trading days, meaning that the put options mature on November 1, 2012. According to Proposition 2.9 and equation (3.1), the costs of the efficient put can be calculated by

$$c\big(\underline{X}_T^P\big) = E\big[e^{-rT}Z_T^{\bar\theta}\underline{X}_T^P\big] = \frac{1}{M_{dist}(\bar\theta)}\int_0^1 e^{\bar\theta F_{dist}^{-1}(1-y)}\big(K - S_0e^{F_{dist}^{-1}(y)}\big)_+\,dy$$

where *dist* is $NIG(\alpha,\beta,\delta T,\mu T)$, $VG(\lambda T,\alpha,\beta,\mu T)$, or $N((\mu-\frac{\sigma^2}{2})T,\sigma^2 T)$.

Using equations (2.5) and (2.11), the costs $c(X_T^P)$ of the vanilla put in the NIG model are given by

$$\begin{aligned} c(X_T^P) &= E_{\bar\theta}\big[e^{-rT}(K-S_T)_+\big] \\ &= e^{-rT}\int_{-\infty}^{\ln(K/S_0)}(K-S_0e^x)Z_T^{\bar\theta}\,d_{NIG(\alpha,\beta,\delta T,\mu T)}(x)\,dx \qquad (3.5) \\ &= Ke^{-rT}F_{NIG(\alpha,\beta+\bar\theta,\delta T,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big) - S_0F_{NIG(\alpha,\beta+\bar\theta+1,\delta T,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big), \end{aligned}$$

and for the VG model one analogously obtains

$$c(X_T^P) = Ke^{-rT}F_{VG(\lambda T,\alpha,\beta+\bar\theta,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big) - S_0F_{VG(\lambda T,\alpha,\beta+\bar\theta+1,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big).$$

In the Samuelson model, $c(X_T^P)$ is, of course, calculated by the well-known Black–Scholes put price formula.

The results show that the savings from choosing the cost-efficient strategies can be quite large: For Allianz, the costs of the efficient put are less

Allianz	$c(X_T^P)$	$c(\underline{X}_T^P)$	**Efficiency loss in %**
NIG	6.4495	5.2825	18.09
VG	6.3681	5.2270	17.92
Normal	6.4324	5.2683	18.10
Volkswagen	$c(X_T^P)$	$c(\underline{X}_T^P)$	**Efficiency loss in %**
NIG	8.0064	4.0871	48.95
VG	7.9765	4.0603	49.10
Normal	7.9909	4.0749	49.01

Table 2: Comparison of the cost of a long put option on Allianz and Volkswagen, resp., and the corresponding cost-efficient payoffs in different Lévy models. Initial stock price, strike, and time to maturity are $S_0 = 93.42$, $K = 98$, $T = 23$ for Allianz and $S_0 = 130.55$, $K = 135$, $T = 23$ for Volkswagen. The other parameters needed for the calculations are taken from Table 1.

than 83% of the price of the plain vanilla put, and in case of Volkswagen the vanilla put is almost twice as expensive as the efficient put. The great differences in the efficiency losses of the Allianz and Volkswagen puts may seem somewhat surprising at first glance because the stock price-strike-ratio $\frac{S_0}{K}$ is roughly the same in both cases (0.953 for Allianz and 0.967 for Volkswagen), but can be explained with help of Theorem 2.10: Since the payoffs X_T^P and $\underline{X}_T^P$ are both bounded by the strike K (confer equation (3.2)), the condition $E_{\bar{\theta}}\big[(X_T^P - \underline{X}_T^P)^2\big] < \infty$ here is trivially fulfilled, so Theorem 2.10 assures that the efficiency loss is increasing in $|\bar{\theta}|$. As can be seen from Table 1, the value of $|\bar{\theta}|$ for Volkswagen is more than 2.5 times as large than that of Allianz, and this is also reflected in the magnitude of the efficiency losses in Table 2. However, for each stock itself the efficiency losses obtained under the different Lévy models are of almost the same size and thus seem to be widely model-independent.

3.2 Call options ($\bar{\theta} > 0$)

Now let us take a closer look at a long call option with strike $K > 0$, maturity $T > 0$, and payoff $X_T^C = (S_T - K)_+ = (S_0 e^{L_T} - K)_+$. As already pointed out before in Example 2.12 ii), X_T^C is monotonically increasing in L_T, hence the long call option is not cost-efficient if $\bar{\theta} > 0$. Its distribution function

$G_C = F_{X_T^C}$ can easily be derived to be

$$G_C(x) = P\big(X_T^C \le x\big) = \begin{cases} 0 & \text{, if } x < 0, \\ F_{L_T}\big(\ln\big(\frac{K+x}{S_0}\big)\big) & \text{, if } x \ge 0. \end{cases} \tag{3.6}$$

The corresponding inverse is given by

$$G_C^{-1}(y) = \big(S_0 e^{F_{L_T}^{-1}(y)} - K\big)_+, \quad y \in (0,1). \tag{3.7}$$

Applying Proposition 2.8 again, for $\bar{\theta} > 0$ the cost-efficient payoff that generates the same distribution G_C as the long call option is given by

$$\underline{X}_T^C = G_C^{-1}(1 - F_{L_T}(L_T)) = \Big(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} - K\Big)_+. \tag{3.8}$$

Similarly, one can show that the short call is inefficient for $\bar{\theta} < 0$ as its payoff function $X_T^{-C} = -(S_0 e^{L_T} - K)_+$ is monotonically decreasing in L_T. The distribution function G_{-C} of the short call payoff is

$$G_{-C}(x) = P\big(X_T^{-C} \le x\big) = \begin{cases} 1 & \text{, if } x \ge 0, \\ 1 - F_{L_T}\big(\ln\big(\frac{K-x}{S_0}\big)\big) & \text{, if } x < 0, \end{cases}$$

and for its inverse one obtains

$$G_{-C}^{-1}(y) = -\big(S_0 e^{F_{L_T}^{-1}(1-y)} - K\big)_+, \quad y \in (0,1),$$

thus the cost-efficient strategy for a short call in the case $\bar{\theta} < 0$ is

$$\underline{X}_T^{-C} = G_{-C}^{-1}(F_{L_T}(L_T)) = -\big(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} - K\big)_+. \tag{3.9}$$

From the preceding equations we also see that the inverse distribution functions of the long and short call fulfill the same equality as in the put case, namely

$$G_C^{-1}(y) = -G_{-C}^{-1}(1-y).$$

Therefore, one can prove in exactly the same way as in Corollary 3.1 that for $\bar{\theta} > 0$ we have, under the general assumptions of Proposition 2.8,

$$\underline{X}_T^C = -\overline{X}_T^{-C} \quad \text{and} \quad c\big(\underline{X}_T^C\big) = -c\big(\overline{X}_T^{-C}\big), \tag{3.10}$$

that is, the payoff function of the cost-efficient long call exactly is the negative payoff of the most-expensive short call, and the same relationship holds for the corresponding costs. If $\bar{\theta} < 0$, one analogously obtains $\underline{X}_T^{-C} = -\overline{X}_T^C$ and $c\big(\underline{X}_T^{-C}\big) = -c\big(\overline{X}_T^C\big)$. Table 3 summarizes the efficiency of long and short put and call positions depending on the sign of the risk-neutral Escher parameter $\bar{\theta}$.

Esscher parameter	$\bar{\theta} < 0$	$\bar{\theta} > 0$
long put	most-expensive	cost-efficient
short put	cost-efficient	most-expensive
long call	cost-efficient	most-expensive
short call	most-expensive	cost-efficient

Table 3: Efficiency of puts and calls depending on the Esscher parameter $\bar{\theta}$.

3.3 Self-quanto calls and puts

A quanto option is a (typically European) option whose payoff is converted into a different currency or numeraire at maturity at a pre-specified rate, called the quanto-factor. In the special case of a self-quanto option the numeraire is the underlying asset price at maturity itself. The payoff of a long self-quanto call with maturity T and strike price K therefore is $X_T^{sqC} = S_T \cdot (S_T - K)_+ = S_0 e^{L_T}(S_0 e^{L_T} - K)_+$ which obviously is monotonically increasing in L_T and thus not cost-efficient if $\bar{\theta} > 0$. To derive the corresponding distribution function $G_{sqC} = F_{X_T^{sqC}}$, observe that the positive solution S_T^* of the quadratic equation $S_T^2 - KS_T = x$, $x > 0$, is given by $S_T^* = \frac{K}{2} + \sqrt{\frac{K^2}{4} + x}$, hence

$$G_{sqC}(x) = P\big(X_T^{sqC} \le x\big) = \begin{cases} 0 & , \text{ if } x < 0, \\ F_{L_T}\left(\ln\left(\frac{\frac{K}{2} + \sqrt{\frac{K^2}{4} + x}}{S_0}\right)\right) & , \text{ if } x \ge 0. \end{cases}$$

The inverse then can easily be shown to equal

$$G_{sqC}^{-1}(y) = S_0 e^{F_{L_T}^{-1}(y)}\big(S_0 e^{F_{L_T}^{-1}(y)} - K\big)_+, \quad y \in (0,1),$$

consequently the cost-efficient strategy for a long self-quanto call in the case $\bar{\theta} > 0$ is, according to Proposition 2.8,

$$\underline{X}_T^{sqC} = G_{sqC}^{-1}(1 - F_{L_T}(L_T)) = S_0 e^{F_{L_T}^{-1}(1 - F_{L_T}(L_T))}\big(S_0 e^{F_{L_T}^{-1}(1 - F_{L_T}(L_T))} - K\big)_+.$$

With analogous arguments as above, the distribution function G_{-sqC} of a short self-quanto call with payoff $X_T^{-sqC} = -S_T \cdot (S_T - K)_+$ can be derived as

$$G_{-sqC}(x) = P\big(X_T^{-sqC} \le x\big) = \begin{cases} 1 & , \text{ if } x \ge 0, \\ 1 - F_{L_T}\left(\ln\left(\frac{\frac{K}{2} + \sqrt{\frac{K^2}{4} - x}}{S_0}\right)\right) & , \text{ if } x < 0, \end{cases}$$

and its inverse is given by

$$G^{-1}_{-sqC}(y) = -S_0 e^{F^{-1}_{L_T}(1-y)}\big(S_0 e^{F^{-1}_{L_T}(1-y)} - K\big)_+.$$

Thus, we again have the symmetry relation $G^{-1}_{sqC}(y) = -G^{-1}_{-sqC}(1-y)$ and conclude in exactly the same way as before that $\underline{X}_T^{sqC} = -\overline{X}_T^{-sqC}$ as well as $c\big(\underline{X}_T^{sqC}\big) = -c\big(\overline{X}_T^{-sqC}\big)$.

In contrast to this, the cost-efficient strategy of a long self-quanto put cannot be derived in closed form along the same lines as above because the payoff $X_T^{sqP} = S_T \cdot (K - S_T)_+ = S_0 e^{L_T}(K - S_0 e^{L_T})_+$ is not monotonic in L_T. Despite this fact, we can still obtain some lower and upper bounds for the costs $c(X_T^{sqP})$ of a long self-quanto put as follows: Using the relations $X_T^{sqP} = S_0 e^{L_T} X_T^P$, $Z_T^{\bar\theta} = \frac{e^{\bar\theta L_T}}{M_{L_T}(\bar\theta)}$, and assuming $\bar\theta + 1 > 0$, we have

$$\begin{aligned} c\big(X_T^{sqP}\big) &= E\big[e^{-rT} Z_T^{\bar\theta} X_T^{sqP}\big] = \frac{M_{L_T}(\bar\theta+1)}{M_{L_T}(\bar\theta)} E\big[e^{-rT} Z_T^{\bar\theta+1} X_T^P\big] \underset{(2.5)}{=} E\big[Z_T^{\bar\theta+1} X_T^P\big] \\ &\le E\big[Z_T^{\bar\theta+1}\overline{X}_T^P\big] = \frac{1}{M_{L_T}(\bar\theta+1)} \int_0^1 e^{(\bar\theta+1)F^{-1}_{L_T}(y)} G_P^{-1}(y)\, dy \end{aligned}$$

where the last equality follows from Proposition 2.9, and G_P^{-1} has been given explicitly in equation (3.1). If $\bar\theta + 1 < 0$, we analogously arrive at the lower bound

$$c\big(X_T^{sqP}\big) \ge E\big[Z_T^{\bar\theta+1}\underline{X}_T^P\big] = \frac{1}{M_{L_T}(\bar\theta+1)} \int_0^1 e^{(\bar\theta+1)F^{-1}_{L_T}(1-y)} G_P^{-1}(1-y)\, dy.$$

Note that the above inequalities are not obtained by solving the original problem of finding the optimal strategies $\underline{X}_T^{sqP}, \overline{X}_T^{sqP} \sim G_{sqP}$ that are co(unter)monotonic to $Z_T^{\bar\theta}$. Therefore, the resulting bounds are not sharp in general, but might serve as useful approximations nevertheless.

3.4 Forwards

The payoff function $X_T^{-F} = K - S_T = K - S_0 e^{L_T}$ of a short forward with delivery price K is strictly decreasing in L_T and thus inefficient if $\bar\theta < 0$. The corresponding distribution function G_{-F} is given by

$$G_{-F}(x) = P(K - S_0 e^{L_T} \le x) = 1 - F_{L_T}\left(\ln\left(\frac{K-x}{S_0}\right)\right)$$

and has the inverse

$$G^{-1}_{-F}(y) = K - S_0 e^{F^{-1}_{L_T}(1-y)}, \quad y \in (0,1).$$

By Proposition 2.8, the cost-efficient strategy for a short forward in the case $\bar{\theta} < 0$ then is

$$\underline{X}_T^{-F} = G_{-F}^{-1}(F_{L_T}(L_T)) = K - S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}. \tag{3.11}$$

Remark 3.2 Observe that the payoff $\underline{X}_T^{-F}$ and hence the costs $c(\underline{X}_T^{-F})$ of the efficient short forward depend on the distribution of L_T and hence on the specific Lévy model one has chosen. In contrast to this, simple no-arbitrage arguments show that the costs $c(X_T^{-F})$ of the standard short forward (if the underlying provides no income during the lifetime of the contract) are given by $c(X_T^{-F}) = Ke^{-rT} - S_0$, and thus, are obviously model-independent.

Recall that the payoff X_T^{-F} of a short forward is equal to the payoff of the sum of a long put and a short call with the same strike K and maturity T, that is,

$$X_T^{-F} = K - S_T = (K - S_T)_+ - (S_T - K)_+.$$

Hence, one can conjecture that the cost-efficient strategy of a short forward may alternatively be derived as the combination of the cost-efficient strategies for a long put and a short call which are both inefficient if $\bar{\theta} < 0$ as seen before. Indeed, from equations (3.2) and (3.9) we have

$$\begin{aligned} \underline{X}_T^{-F} &= K - S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} \\ &= \left(K - S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}\right)_+ - \left(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} - K\right)_+ \\ &= \underline{X}_T^P + \underline{X}_T^{-C} \end{aligned} \tag{3.12}$$

which proves the above conjecture.

Analogously, a long forward is inefficient if $\bar{\theta} > 0$. Its payoff X_T^F corresponds to the sum of the payoffs of a long call and a short put:

$$X_T^F = S_T - K = (S_T - K)_+ - (K - S_T)_+. \tag{3.13}$$

Here, one also obtains that the payoff of a cost-efficient long forward

$$\underline{X}_T^F = G_F^{-1}(1 - F_{L_T}(L_T)) = S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} - K \tag{3.14}$$

corresponds to the sum of a cost-efficient long call and a cost-efficient short put.

The decomposition of the cost-efficient strategy of a forward into the sum of cost-efficient strategies for a call and a put suggests that there might exist some cost-efficient analogue to the put-call parity. As is known, the latter uses the representation (3.13) to conclude that the difference

$c(X_T^C) - c(X_T^P)$ of the long call and put prices must equal the long forward price $c(X_T^F) = S_0 - Ke^{-rT}$. Therefore, the prices of a long call and a long put can be derived from each other by just adding resp. substracting the corresponding forward price. As shown in equation (3.12), the relation (3.13) analogously also holds for the cost-efficient strategies $\underline{X}_T^C$, $\underline{X}_T^{-P}$, and $\underline{X}_T^F$, so one might conjecture that, for example, the price of a cost-efficient long put can similarly be obtained as the difference of the prices of an efficient long call and an efficient long forward. However, it can easily be seen that this cannot be true: This line of argumentation essentially relies on the fact that the payoffs and costs of short positions are just the negative payoffs resp. costs of the corresponding long positions, but as we have seen above in Corollary 3.1, this simple relationship does not apply to cost-efficient payoffs anymore. Altering between long and short positions here means to switch between cost-efficient and most-expensive payoffs, so the correct put-call parity within this framework is given by

Proposition 3.2 (Cost-efficient put-call parity) *The cost-efficient long forward payoff $\underline{X}_T^F$ allows the decomposition $\underline{X}_T^F = \underline{X}_T^C + \underline{X}_T^{-P}$ which implies $\underline{X}_T^C = \underline{X}_T^F + \overline{X}_T^P$, and thus, $c(\underline{X}_T^C) = c(\underline{X}_T^F) + c(\overline{X}_T^P)$.*

For the price of a cost-efficient long put one analogously obtains

$$c(\underline{X}_T^P) = c(\underline{X}_T^{-F}) + c(\overline{X}_T^C). \tag{3.15}$$

Remember that for $\bar{\theta} < 0$ we have $X_T^C = \underline{X}_T^C$, $X_T^F = \underline{X}_T^F$, and $X_T^P = \overline{X}_T^P$, whereas for $\bar{\theta} > 0$ the standard payoffs $X_T^P = \underline{X}_T^P$, $X_T^{-F} = \underline{X}_T^{-F}$, and $X_T^C = \overline{X}_T^C$ are already cost-efficient, resp. most-expensive.

3.5 Bull and bear spreads

A bull spread is a combination of a long call C_1 with strike $K_1 > 0$ and a short call $-C_2$ with strike $K_2 > K_1$. The payoff is given by

$$X_T^{bull} = \big(S_0 e^{L_T} - K_1\big)_+ - \big(S_0 e^{L_T} - K_2\big)_+,$$

and thus is increasing in L_T. Hence, the bull spread is not cost-efficient if $\bar{\theta} > 0$. Its distribution function is

$$G_{bull}(x) = \begin{cases} 0 & , \text{ if } x < 0, \\ F_{L_T}\big(\ln\big(\frac{K_1+x}{S_0}\big)\big) & , \text{ if } 0 \le x < K_2 - K_1, \\ 1 & , \text{ if } x \ge K_2 - K_1, \end{cases}$$

and the corresponding inverse can be represented by

$$G_{bull}^{-1}(y) = \big(S_0 e^{F_{L_T}^{-1}(y)} - K_1\big)_+ - \big(S_0 e^{F_{L_T}^{-1}(y)} - K_2\big)_+.$$

Note that for $x < K_2 - K_1$, the distribution function $G_{bull}(x)$ coincides with that of the long call C_1, therefore it is not surprising that the first summand of the inverse $G_{bull}^{-1}(y)$ is equal to $G_{C_1}^{-1}(y)$. The second summand here is necessary to ensure that the quantile function takes only values in the range $[0, K_2 - K_1]$ of X_T^{bull}. If $\bar{\theta} > 0$, the cost-efficient strategy corresponding to such a bull spread then is

$$\begin{aligned}\underline{X}_T^{bull} &= G_{bull}^{-1}(1 - F_{L_T}(L_T)) \\ &= \left(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} - K_1\right)_+ - \left(S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))} - K_2\right)_+ \\ &= \underline{X}_T^{C_1} - \underline{X}_T^{C_2} = \underline{X}_T^{C_1} + \overline{X}_T^{-C_2} \qquad (3.16)\end{aligned}$$

where the last equalities follow from equations (3.8), (3.9), and (3.10). Hence, the efficient bull spread payoff $\underline{X}_T^{bull}$ is eqivalent to a long position in an efficient call C_1 and a short position in a most-expensive call C_2, which again is a very intuitive result (see also the discussion preceding Corollary 3.1).

A bear spread is a combination of a short put with strike $K_1 > 0$ and a long put with strike $K_2 > K_1$. Its payoff thus is

$$X_T^{bear} = \left(K_2 - S_0 e^{L_T}\right)_+ - \left(K_1 - S_0 e^{L_T}\right)_+$$

which obviously is decreasing in L_T and thus inefficient if $\bar{\theta} < 0$. Similarly to the bull spread, the inverse G_{bear}^{-1} of the distribution function of X_T^{bear} equals

$$G_{bear}^{-1}(y) = \left(K_2 - S_0 e^{F_{L_T}^{-1}(1-y)}\right)_+ - \left(K_1 - S_0 e^{F_{L_T}^{-1}(1-y)}\right)_+$$

from which we derive the cost-efficient payoff of the bear spread for $\bar{\theta} < 0$ as

$$\begin{aligned}\underline{X}_T^{bear} &= \left(K_2 - S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}\right)_+ - \left(K_1 - S_0 e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}\right)_+ \\ &= \underline{X}_T^{P_2} - \underline{X}_T^{P_1} = \underline{X}_T^{P_2} + \overline{X}_T^{-P_1} \qquad (3.17)\end{aligned}$$

which corresponds to the sum of an efficient long put P_2 with strike K_2 and a most-expensive short put $-P_1$ with strike K_1.

From the above examples, one may have the impression that the cost-efficient strategy for any combination of long and short puts or calls can easily be obtained by just replacing the long positions by their cost-efficient and the short positions by their most-expensive counterparts. However, this is not true in general as the following counterexample shows: Consider a butterfly which is the combination of two long calls C_3 and C_1 with strikes

$K_3 > K_1 > 0$, and two short calls $-C_2$ with strike $K_2 = \frac{K_1+K_3}{2}$. The payoff X_T^{bfly} of a butterfly spread thus is given by

$$X_T^{\mathit{bfly}} = \big(S_0 e^{L_T} - K_1\big)_+ + \big(S_0 e^{L_T} - K_3\big)_+ - 2\big(S_0 e^{L_T} - K_2\big)_+,$$

and the corresponding distribution function G_{bfly} can be shown to equal

$$G_{\mathit{bfly}}(x) = \begin{cases} 0 & , \text{ if } x < 0, \\ F_{L_T}\big(\ln\big(\frac{K_1+x}{S_0}\big)\big) + 1 - F_{L_T}\big(\ln\big(\frac{K_3-x}{S_0}\big)\big) & , \text{ if } 0 \le x < \frac{K_3-K_1}{2}, \\ 1 & , \text{ if } x \ge \frac{K_3-K_1}{2}. \end{cases}$$

Here the distribution function has a more complex form because the payoff X_T^{bfly} is not monotonic in L_T, and it can easily be checked that the inverse G_{bfly}^{-1} does not admit a representation in form of a sum of $G_{C_1}^{-1}$, $G_{C_3}^{-1}$, and $G_{-C_2}^{-1}$. Therefore, the relation $\underline{X}_T^{\mathit{bfly}} = \underline{X}_T^{C_1} + \underline{X}_T^{C_3} + 2\overline{X}_T^{-C_2}$ cannot be valid either.

4 Delta hedging of cost-efficient strategies in Lévy models

In the previous section we provided a semi-explicit formula for the costs of a cost-efficient strategy which is valuable for many financial applications since it can be easily evaluated numerically. For practitioners, however, this formula might still be unsatisfying unless an explicit hedging strategy for the cost-efficient payoff exists. In this section we want to close this gap and first provide some formulas of possible hedging strategies for efficient puts and calls which we then apply to hedge the efficient puts on Allianz and Volkswagen discussed in Section 3.1. In the following we focus on developing a formula for the delta hedge, i.e., the derivative of the cost of a strategy with respect to the underlying, for the cost-efficient payoff of a put resp. call option. If the underlying asset is traded sufficiently liquid in the market, delta hedging probably is one of the simplest, but nevertheless fairly effective ways to cover a risky position and is therefore widely used in practice.

4.1 Theoretical results

Consider the payoffs $X_T^P = (K - S_T)_+$ and $X_T^C = (S_T - K)_+$ of a put resp. call option with strike $K > 0$ and maturity $T > 0$. As mentioned in Section 3, the payoff X_T^P becomes inefficient for $\bar{\theta} < 0$ while the payoff X_T^C is inefficient if $\bar{\theta} > 0$. Inserting the inverse distribution functions G_P^{-1} and G_C^{-1} which we derived in equations (3.1) and (3.7) into the corresponding formulas of Proposition 2.9, we obtain the price of the cost-efficient put option as

$$c(\underline{X}_T^P) = \frac{1}{M_{L_T}(\bar{\theta})} \int_0^1 e^{\bar{\theta} F_{L_T}^{-1}(1-y) - rT} \left(K - S_0 e^{F_{L_T}^{-1}(y)}\right)_+ dy, \quad \bar{\theta} < 0, \qquad (4.1)$$

and the price of a cost-efficient call option is given by

$$c(\underline{X}_T^C) = \frac{1}{M_{L_T}(\bar{\theta})} \int_0^1 e^{\bar{\theta} F_{L_T}^{-1}(y) - rT} \left(S_0 e^{F_{L_T}^{-1}(1-y)} - K\right)_+ dy, \quad \bar{\theta} > 0. \qquad (4.2)$$

The following theorem provides explicit formulas for the derivatives of $c(\underline{X}_T^P)$ and $c(\underline{X}_T^C)$ with respect to the underlying S_0.

Theorem 4.1 (Deltas for cost-efficient puts and calls) *Let $(L_t)_{t \geq 0}$ be a Lévy process with a continuous and strictly increasing distribution F_{L_T} at maturity $T > 0$, and assume that a solution $\bar{\theta}$ of (2.5) exists.*

i) If $\bar{\theta} < 0$, the delta $\underline{\Delta}_T^P$ of the cost-efficient long put $\underline{X}_T^P$ is given by

$$\underline{\Delta}_T^P = -\frac{1}{M_{L_T}(\bar{\theta})} \int_0^{F_{L_T}\left(\ln\left(\frac{K}{S_0}\right)\right)} e^{\bar{\theta} F_{L_T}^{-1}(1-y) + F_{L_T}^{-1}(y) - rT} dy. \qquad (4.3)$$

ii) If $\bar{\theta} > 0$, the delta $\underline{\Delta}_T^C$ of the cost-efficient long call $\underline{X}_T^C$ is given by

$$\underline{\Delta}_T^C = \frac{1}{M_{L_T}(\bar{\theta})} \int_0^{1 - F_{L_T}\left(\ln\left(\frac{K}{S_0}\right)\right)} e^{\bar{\theta} F_{L_T}^{-1}(y) + F_{L_T}^{-1}(1-y) - rT} dy. \qquad (4.4)$$

Proof: i) The price $c(\underline{X}_T^P)$ of the cost-efficient long put has been given in equation (4.1). It can be easily seen that the integrand is integrable with respect to y for all $S_0 \geq 0$ since, due to Proposition 2.9, $c(\underline{X}_T^P)$ is always bounded from above by the price $c(X_T^P)$ of the standard (inefficient) long put which obviously is finite for all $S_0 \in \mathbb{R}_+$. Moreover, the function

$$f(S_0, y) : \mathbb{R} \times [0,1] \to \mathbb{R}_+, \qquad f(S_0, y) = e^{\bar{\theta} F_{L_T}^{-1}(1-y) - rT} \left(K - S_0 e^{F_{L_T}^{-1}(y)}\right)_+$$

is differentiable in S_0 for all $y \in [0,1]$ (apart from the point $S_0 = K e^{-F_{L_T}^{-1}(y)}$, but since the left- and right-hand side derivatives are bounded, this can be neglected here), and the partial derivative is

$$\frac{\partial}{\partial S_0} f(S_0, y) = e^{\bar{\theta} F_{L_T}^{-1}(1-y) - rT} \left(-e^{F_{L_T}^{-1}(y)}\right) \mathbb{1}_{[0, F_{L_T}(\ln(K/S_0))]}(y).$$

Its absolute value is bounded by the integrable function

$$g(y) = \frac{K}{S_0}\, e^{\bar{\theta}F_{L_T}^{-1}(1-y)-rT}.$$

For the integrability of $g(y)$, observe that

$$\int_0^1 g(y)\, dy = \frac{K}{S_0}\int_0^1 e^{\bar{\theta}F_{L_T}^{-1}(1-y)-rT} dy = \frac{K}{S_0}\int_0^1 e^{\bar{\theta}F_{L_T}^{-1}(z)-rT}\, dz$$

$$= \frac{K}{S_0}\int_{-\infty}^{+\infty} e^{\bar{\theta}x-rT} f_{L_T}(x)\, dx = \frac{Ke^{-rT}}{S_0}\, M_{L_T}(\bar{\theta}) < \infty,$$

where f_{L_T} denotes the density of L_T which exists and is strictly positive on $\mathbb{R}$ due to our assumptions on F_{L_T}. Hence, we can interchange differentiation and integration and obtain

$$\underline{\Delta}_T^P = \frac{\partial}{\partial S_0} c(\underline{X}_T^P) = \frac{1}{M_{L_T}(\bar{\theta})}\int_0^1 \frac{\partial}{\partial S_0} f(S_0, y)\, dy$$

$$= -\frac{1}{M_{L_T}(\bar{\theta})}\int_0^{F_{L_T}(\ln(K/S_0))} e^{\bar{\theta}F_{L_T}^{-1}(1-y)+F_{L_T}^{-1}(y)-rT} dy.$$

ii) For the cost-efficient call whose price has been given in (4.2) we consider the function

$$\tilde{f}(S_0, y) :\ \mathbb{R}\times[0,1] \to \mathbb{R}_+, \qquad \tilde{f}(S_0, y) = e^{\bar{\theta}F_{L_T}^{-1}(y)-rT}\big(S_0 e^{F_{L_T}^{-1}(1-y)} - K\big)_+$$

which is integrable with respect to y for all $S_0 \geq 0$ (this follows analogously as in the put case). Further, $\tilde{f}(S_0, y)$ is differentiable in S_0 for all $y \in [0,1]$ (apart from $S_0 = Ke^{-F_{L_T}^{-1}(1-y)}$ which again is negligible here), and we have

$$\frac{\partial}{\partial S_0}\tilde{f}(S_0, y) = e^{\bar{\theta}F_{L_T}^{-1}(y)-rT}\cdot e^{F_{L_T}^{-1}(1-y)}\, \mathbb{1}_{[0,1-F_{L_T}(\ln(K/S_0))]}(y) \geq 0.$$

Clearly, the integrability of $\tilde{f}(S_0, y)$ with respect to y readily transfers to $\frac{\partial}{\partial S_0}\tilde{f}(S_0, y)$, thus we can again interchange differentiation and integration and obtain

$$\underline{\Delta}_T^C = \frac{\partial}{\partial S_0} c(\underline{X}_T^C) = \frac{1}{M_{L_T}(\bar{\theta})}\int_0^1 \frac{\partial}{\partial S_0}\tilde{f}(S_0, y)\, dy$$

$$= \frac{1}{M_{L_T}(\bar{\theta})}\int_0^{1-F_{L_T}(\ln(K/S_0))} e^{\bar{\theta}F_{L_T}^{-1}(y)+F_{L_T}^{-1}(1-y)-rT} dy.$$

□

Equations (4.3) and (4.4) of the previous theorem especially entail that the deltas of efficient puts and calls are always negative resp. positive, that is, analogous to its classical counterparts one has to go short in the underlying to hedge a cost-efficient put, and hedging of an efficient call requires a long position in the underlying. This might be somewhat surprising at first because Figure 2 has shown that the payoff of an efficient put is reversed to that of a vanilla put and bears some similarities to the payoff of a vanilla call, and a similar observation can be made when comparing the payoffs of standard and efficient calls. Thus, one might expect that $\underline{\Delta}_T^P$ and $\underline{\Delta}_T^C$ just have the opposite signs of the deltas Δ_T^P and Δ_T^C of the vanilla put resp. call, but the following consideration shows that this line of argumentation is misleading: As already pointed out in Remark 3.1, a cost-efficient put must provide the same payoff-distribution as a vanilla put at any time t within the lifetime $[0, T]$ of the contract. But since the asset price S_T is a known constant at maturity, the payoff-distribution of the standard put at time T is the degenerate distribution (unit mass) located at $(K - S_T)_+$, implying that the terminal payoffs of both the standard and the cost-efficient put must coincide. (Using the fact that $L_0 = 0$ almost surely for every Lévy process L, one also easily obtains that $\underline{X}_{T-t}^P = \left(K - S_t e^{F_{L_{T-t}}^{-1}(1-F_{L_{T-t}}(L_{T-t}))}\right)_+ \to (K - S_T)_+$ for $t \to T$.) Hence, it is intuitively obvious that—at least close to maturity—the hedging strategies of the standard and the efficient put should be fairly similar and therefore $\underline{\Delta}_{T-t}^P$ and Δ_{T-t}^P should have the same sign. Theorem 4.1 assures that this holds true for all $t \in [0, T]$.

However, because the prices of the efficient calls and puts are smaller than those of their standard counterparts, one then could at least expect that the absolute values $|\underline{\Delta}_T^P|$ and $|\underline{\Delta}_T^C|$ are also smaller than $|\Delta_T^P|$ and $|\Delta_T^C|$, respectively. The next theorem shows that such a relation indeed is always fulfilled for the call deltas and in most cases also for the put deltas.

Theorem 4.2 (Comparison of deltas) *Let $(L_t)_{t\geq 0}$ be a Lévy process with a continuous and strictly increasing distribution F_{L_T} at maturity $T > 0$, and assume that a solution $\bar{\theta}$ of* (2.5) *exists.*

a) For cost-efficient and vanilla calls, we have the following relations: If $\bar{\theta} > 0$, then $0 \leq \underline{\Delta}_T^C \leq \Delta_T^C$. For $\bar{\theta} < 0$ we have $\underline{\Delta}_T^C = \Delta_T^C$.

b) In the put case, we have $\underline{\Delta}_T^P = \Delta_T^P$ for $\bar{\theta} > 0$. If $\bar{\theta} < 0$ and $F_{L_T}\big(\ln\big(\frac{K}{S_0}\big)\big) \leq q^*$ where $q^* \in (0.5, 1]$ is the unique positive root of the function $D_P : [0, 1] \to \mathbb{R}$,

$$D_P(q) = \frac{1}{M_{L_T}(\bar{\theta})} \int_0^q e^{\bar{\theta} F_{L_T}^{-1}(y) + F_{L_T}^{-1}(y)} - e^{\bar{\theta} F_{L_T}^{-1}(1-y) + F_{L_T}^{-1}(y)}\, dy,$$

then the relation $\Delta_T^P \leq \underline{\Delta}_T^P \leq 0$ also holds.

Proof: a) Since the vanilla and the efficient call coincide for $\bar\theta < 0$, the equation $\underline{\Delta}_T^C = \Delta_T^C$ is immediately obvious, thus we only have to consider the case $\bar\theta > 0$. Because the vanilla call is most-expensive for $\bar\theta > 0$, we can combine Proposition 2.9 and equation (3.7) to represent its price by

$$c(X_T^C) = \frac{1}{M_{L_T}(\bar\theta)} \int_0^1 e^{\bar\theta F_{L_T}^{-1}(y) - rT} \big(S_0 e^{F_{L_T}^{-1}(y)} - K\big)_+ \, dy$$

from which one can derive completely analogously as in the proof of Theorem 4.1 the following formula for the corresponding delta:

$$\begin{aligned}
\Delta_T^C &= \frac{1}{M_{L_T}(\bar\theta)} \int_{F_{L_T}\left(\ln\left(\frac{K}{S_0}\right)\right)}^{1} e^{\bar\theta F_{L_T}^{-1}(y) + F_{L_T}^{-1}(y) - rT} \, dy \\
&= \frac{1}{M_{L_T}(\bar\theta)} \int_0^{1 - F_{L_T}\left(\ln\left(\frac{K}{S_0}\right)\right)} e^{\bar\theta F_{L_T}^{-1}(1-y) + F_{L_T}^{-1}(1-y) - rT} \, dy \qquad (4.5)
\end{aligned}$$

Because $\underline{\Delta}_T^C \geq 0$ by Theorem 4.1, the assertion of the theorem is proven if we can show that $\Delta_T^C - \underline{\Delta}_T^C \geq 0$. Comparing equations (4.4) and (4.5), the latter inequality obviously is equivalent to the statement that the function $D_C : [0,1] \to \mathbb{R}$, defined by

$$D_C(q) = \frac{1}{M_{L_T}(\bar\theta)} \int_0^q e^{\bar\theta F_{L_T}^{-1}(1-y) + F_{L_T}^{-1}(1-y)} - e^{\bar\theta F_{L_T}^{-1}(y) + F_{L_T}^{-1}(1-y)} \, dy,$$

is nonnegative for all $q \in [0,1]$. We have $D_C(0) = 0$ and

$$\begin{aligned}
D_C(1) &= \frac{1}{M_{L_T}(\bar\theta)} \int_0^1 e^{\bar\theta F_{L_T}^{-1}(1-y) + F_{L_T}^{-1}(1-y)} - e^{\bar\theta F_{L_T}^{-1}(y) + F_{L_T}^{-1}(1-y)} \, dy \\
&= \int_{-\infty}^{\infty} \frac{e^{\bar\theta z}}{M_{L_T}(\bar\theta)} \, e^z f_{L_T}(z) \, dz - \int_{-\infty}^{\infty} \frac{e^{\bar\theta z}}{M_{L_T}(\bar\theta)} \, e^{F_{L_T}^{-1}(1 - F_{L_T}(z))} f_{L_T}(z) \, dz \\
&= E\left[Z_T^{\bar\theta} \, \frac{S_T}{S_0}\right] - E\big[Z_T^{\bar\theta} \, e^{F_{L_T}^{-1}(1 - F_{L_T}(L_T))}\big] \geq 0
\end{aligned}$$

because $\frac{S_T}{S_0} = e^{L_T} \overset{d}{=} e^{F_{L_T}^{-1}(1 - F_{L_T}(L_T))}$, but $Z_T^{\bar\theta}$, e^{L_T} are comonotonic and $Z_T^{\bar\theta}$, $e^{F_{L_T}^{-1}(1 - F_{L_T}(L_T))}$ are countermonotonic for $\bar\theta > 0$. Moreover,

$$D_C'(q) = \frac{1}{M_{L_T}(\bar\theta)} \left[e^{\bar\theta F_{L_T}^{-1}(1-q) + F_{L_T}^{-1}(1-q)} - e^{\bar\theta F_{L_T}^{-1}(q) + F_{L_T}^{-1}(1-q)}\right]$$

from which we conclude

$$D'_C(q) = 0 \quad\Longleftrightarrow\quad e^{\bar\theta F_{L_T}^{-1}(1-q)} = e^{\bar\theta F_{L_T}^{-1}(q)} \quad\Longleftrightarrow\quad q = 0.5.$$

The assumptions on F_{L_T} imply that $F_{L_T}^{-1}(q)$ is strictly increasing as well, so the above calculations further show that $D'_C(q) \geq 0$ for $q \leq 0.5$ and $D'_C(q) \leq 0$ for $q \geq 0.5$. Hence, the function $D_C(q)$ is increasing on $[0, 0.5]$ and decreasing on $[0.5, 1]$ with boundary values $D_C(0) = 0$ and $D_C(1) \geq 0$ which yields that $D_C(q) \geq 0$ for all $q \in [0, 1]$ and thus $\Delta_T^C - \underline{\Delta}_T^C \geq 0$.

b) The equality $\underline{\Delta}_T^P = \Delta_T^P$ for $\bar\theta > 0$ again follows from the fact that vanilla and efficient put coincide in this case, therefore we assume $\bar\theta < 0$ in the following. Then the vanilla put is most-expensive, and combining Proposition 2.9 and equation (3.1) allows to represent its price as

$$c(X_T^P) = \frac{1}{M_{L_T}(\bar\theta)} \int_0^1 e^{\bar\theta F_{L_T}^{-1}(1-y) - rT} \big(K - S_0 e^{F_{L_T}^{-1}(1-y)}\big)_+ \, dy$$

from which the delta can be derived as

$$\begin{aligned}\Delta_T^P &= -\frac{1}{M_{L_T}(\bar\theta)} \int_{1-F_{L_T}\left(\ln\left(\frac{K}{S_0}\right)\right)}^{1} e^{\bar\theta F_{L_T}^{-1}(1-y) + F_{L_T}^{-1}(1-y) - rT} \, dy \\ &= -\frac{1}{M_{L_T}(\bar\theta)} \int_0^{F_{L_T}\left(\ln\left(\frac{K}{S_0}\right)\right)} e^{\bar\theta F_{L_T}^{-1}(y) + F_{L_T}^{-1}(y) - rT} \, dy. \qquad (4.6)\end{aligned}$$

Because $\underline{\Delta}_T^P, \Delta_T^P \leq 0$, the assertion of the theorem is equivalent to $\underline{\Delta}_T^P - \Delta_T^P \geq 0$. Analogously as in the call case we see by comparing equations (4.3) and (4.6) that for given values K and S_0 we have $\underline{\Delta}_T^P - \Delta_T^P \geq 0$ if and only if $D_P\big[F_{L_T}\big(\ln\big(\frac{K}{S_0}\big)\big)\big] \geq 0$, where the function $D_P(q) : [0, 1] \to \mathbb{R}$ is defined by

$$D_P(q) = \frac{1}{M_{L_T}(\bar\theta)} \int_0^q e^{\bar\theta F_{L_T}^{-1}(y) + F_{L_T}^{-1}(y)} - e^{\bar\theta F_{L_T}^{-1}(1-y) + F_{L_T}^{-1}(y)} \, dy.$$

We have $D_P(0) = 0$ and calculate, similarly as before, that

$$D_P(1) = E\left[Z_T^{\bar\theta} \frac{S_T}{S_0}\right] - E\big[Z_T^{\bar\theta} \, e^{F_{L_T}^{-1}(1 - F_{L_T}(L_T))}\big] \leq 0$$

because for $\bar\theta < 0$ now $Z_T^{\bar\theta}$, e^{L_T} are countermonotonic and $Z_T^{\bar\theta}$, $e^{F_{L_T}^{-1}(1-F_{L_T}(L_T))}$ are comonotonic. Further,

$$D'_P(q) = \frac{1}{M_{L_T}(\bar\theta)} \big[e^{\bar\theta F_{L_T}^{-1}(q) + F_{L_T}^{-1}(q)} - e^{\bar\theta F_{L_T}^{-1}(1-q) + F_{L_T}^{-1}(q)}\big]$$

and hence

$$D'_P(q) = 0 \iff e^{\bar{\theta} F^{-1}_{L_T}(1-q)} = e^{\bar{\theta} F^{-1}_{L_T}(q)} \iff q = 0.5.$$

Since $F^{-1}_{L_T}(q)$ is strictly increasing and $\bar{\theta} < 0$, we see that $D'_P(q) \geq 0$ for $q \leq 0.5$ and $D'_P(q) \leq 0$ for $q \geq 0.5$, consequently the function $D_P(q)$ has a positive maximum in $q = 0.5$ and is strictly decreasing on $(0.5, 1]$. The fact that $D_P(1) \leq 0$ then implies the existence of a unique $q^* \in (0.5, 1]$ with $D_P(q^*) = 0$ and $D_P(q) \geq 0$ for all $q \in [0, q^*]$. This proves the assertion. □

Remark 4.1 Observe again that the condition in Theorem 4.2 b) is time-dependent, thus even if $\Delta^P_T \leq \underline{\Delta}^P_T$ holds at the initial time $t = 0$, this does not necessarily mean that $\Delta^P_{T-t} \leq \underline{\Delta}^P_{T-t}$ for all $t \in (0, T]$. However, in practical examples this inequality typically holds throughout the lifetime of the contract as the subsequent examples show. The fact that $\underline{\Delta}^P_T$ and $\underline{\Delta}^C_T$ are smaller than their counterparts of the corresponding vanilla options also implies that the prices of cost-efficient puts and calls react less sensitive to changes in value of the underlying and thus may allow for more efficient hedging strategies of such contracts. The results of the next subsection indicate that this indeed is the case.

4.2 Application to real market data

In the following we illustrate the theoretical findings by some practical examples for the put case which continue the calculations in Section 3.1. More specifically, we consider the evolution of vanilla and cost-efficient puts on the Allianz and the Volkswagen stock which are assumed to be issued on October 1, 2012, and to mature on November 1, 2012. Figures 4 and 5 show the prices of the Allianz stock and the corresponding puts with strike $K = 98$ within the aforementioned time period as well as the values of the deltas associated to both puts. Here, all calculations are based on the NIG model; the NIG parameters for Allianz can be found in Table 1. The deltas $\underline{\Delta}^P_{T-t}$ of the efficient put were calculated using equation (4.3) from Theorem 4.1, and an explicit formula for their counterparts Δ^P_{T-t} of the vanilla put in the NIG model can be easily derived from equation (3.5): Observing that

$$\begin{aligned} d_{NIG(\alpha,\beta+\bar{\theta}+1,\delta T,\mu T)}(x) &\underset{(2.11)}{=} \frac{e^{(\bar{\theta}+1)x}}{M_{L_T}(\bar{\theta}+1)} \, d_{NIG(\alpha,\beta,\delta T,\mu T)}(x) \\ &\underset{(2.5)}{=} \frac{e^x}{e^{rT}} \, \frac{e^{\bar{\theta}x}}{M_{L_T}(\bar{\theta})} \, d_{NIG(\alpha,\beta,\delta T,\mu T)}(x) \\ &\underset{(2.11)}{=} \frac{e^x}{e^{rT}} \, d_{NIG(\alpha,\beta+\bar{\theta},\delta T,\mu T)}(x), \end{aligned}$$

we obtain that

$$\begin{aligned}\frac{\partial c(X_T^P)}{\partial S_0} &= -\frac{Ke^{-rT}}{S_0}\, d_{NIG(\alpha,\beta+\bar\theta,\delta T,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big) - F_{NIG(\alpha,\beta+\bar\theta+1,\delta T,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big)\\ &\qquad + d_{NIG(\alpha,\beta+\bar\theta+1,\delta T,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big)\\ &= -\frac{Ke^{-rT}}{S_0}\, d_{NIG(\alpha,\beta+\bar\theta,\delta T,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big) - F_{NIG(\alpha,\beta+\bar\theta+1,\delta T,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big)\\ &\qquad + \frac{Ke^{-rT}}{S_0}\, d_{NIG(\alpha,\beta+\bar\theta,\delta T,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big)\\ &= -F_{NIG(\alpha,\beta+\bar\theta+1,\delta T,\mu T)}\big(\ln\big(\tfrac{K}{S_0}\big)\big).\end{aligned}$$

As is obvious from Figure 4, the price of the cost-efficient put converges to that of the vanilla put when the time to maturity tends to zero. This reflects the fact that the payoffs of both puts coincide at the end as pointed out before.

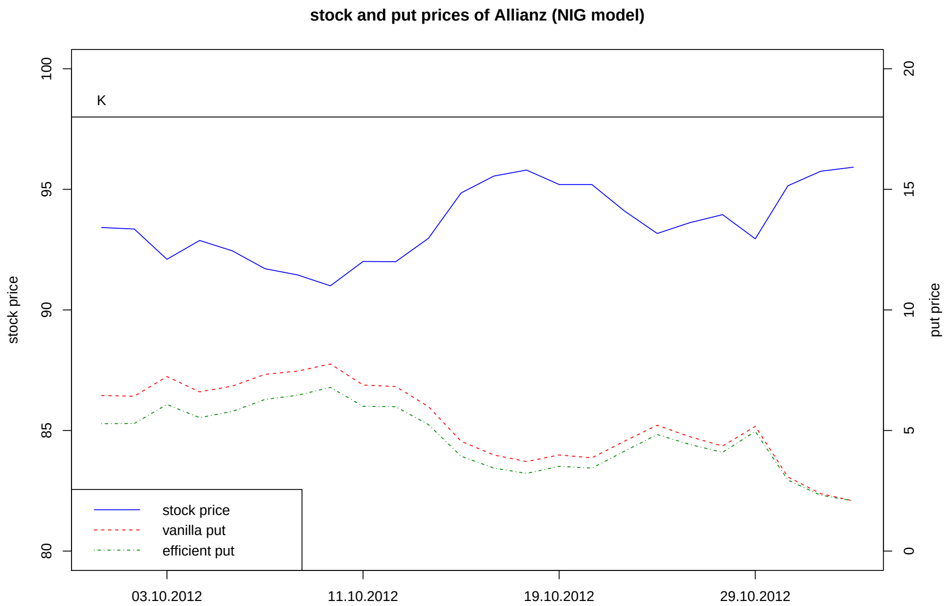

Figure 4: Evolution of the stock price of Allianz from October 1, 2012, to November 1, 2012, and the prices of the associated vanilla and efficient puts in the NIG model with strike $K = 98$ and maturity date November 1, 2012. The NIG parameters used to calculate the put prices can be found in Table 1.

The deltas associated to the Allianz puts fulfill the relation $\underline{\Delta}_{T-t}^P - \Delta_{T-t}^P \geq 0$ for all $t \in [0, T]$. Because the values of the deltas at maturity are always trivial (-1 if the put ends in the money, and 0 otherwise) and not relevant for hedging purposes anymore, Figure 5 only shows the deltas up to one day to maturity, that is, from October 1, 2012, to October 31, 2012.

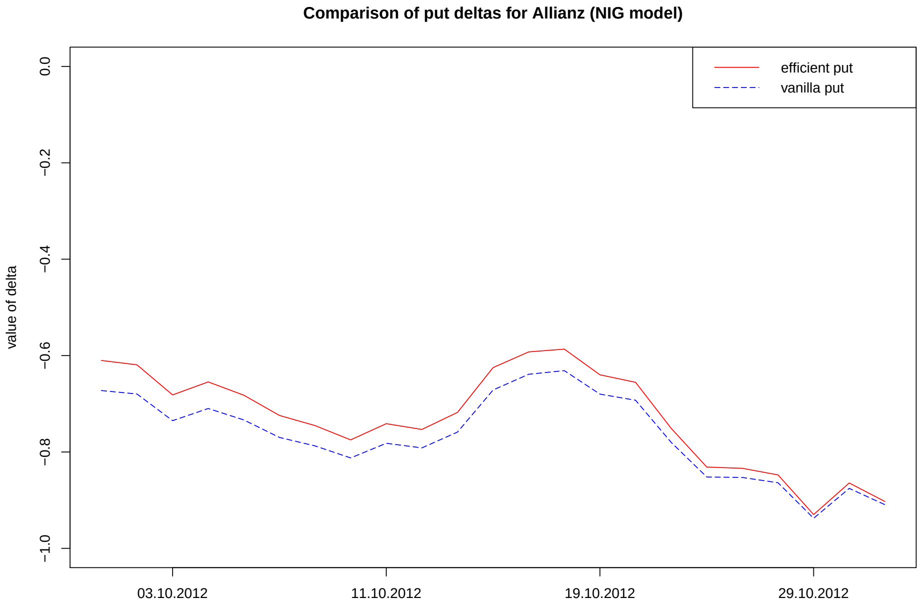

Figure 5: Comparison of the deltas corresponding to the vanilla and efficient put on Allianz within the NIG model shown in Figure 4.

The results obtained for the other two Lévy models (Normal and VG) look quite similar and therefore are not plotted here separately. This is also in line with our previous estimations and calculations. Since the risk-neutral Esscher parameter roughly was of the same size for all three models (see Table 1) and also the put prices and efficiency losses in Table 2 were almost identical, one should not expect greater differences here.

The next two pictures show the evolution of the prices of the Volkswagen stock and the cost-efficient and vanilla puts on it with strike $K = 135$ as well as the corresponding deltas. Again, the results do not differ much between all three Lévy models under consideration, thus we only show the plots for the VG case only. The delta of the vanilla put in this model can be derived analogously as above to be

$$\Delta_T^P = \frac{\partial c(X_T^P)}{\partial S_0} = -F_{VG(\lambda T, \alpha, \beta + \bar{\theta} + 1, \mu T)}\left(\ln\left(\tfrac{K}{S_0}\right)\right).$$

Note that in this example both puts expire worthless, therefore their prices as well as the deltas both tend to zero at maturity.

However, computing the put deltas is only one side of the coin, market participants will surely be more interested in how well the hedging strategies based on them work in practice. The NIG and VG models are incomplete from the very beginning, so one cannot expect perfect hedging there,

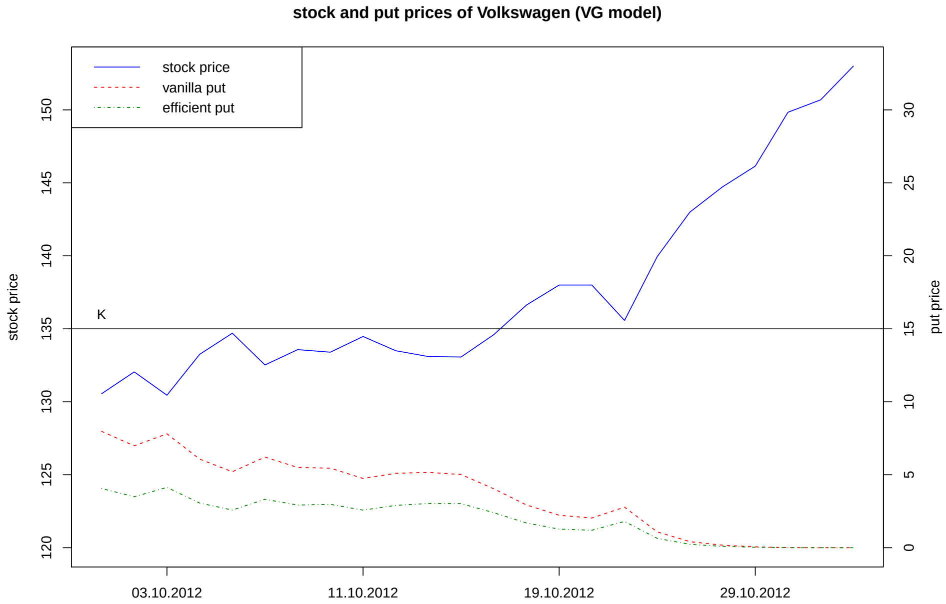

Figure 6: Evolution of the stock price of Volkswagen from October 1, 2012, to November 1, 2012, and the prices of the associated vanilla and efficient puts in the VG model with strike $K = 135$ and maturity date November 1, 2012. The VG parameters used to calculate the put prices can be found in Table 1.

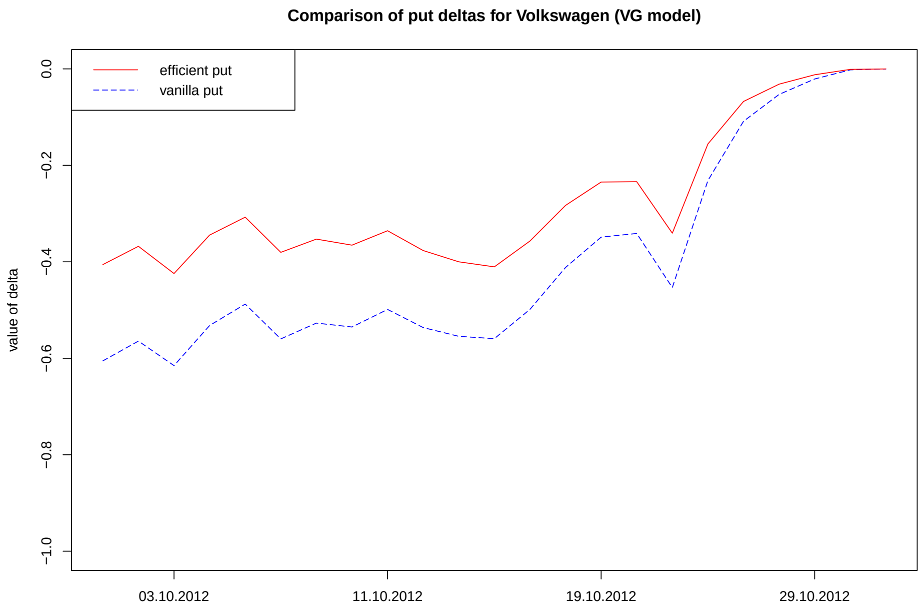

Figure 7: Comparison of the Deltas corresponding to the vanilla and efficient put on Volkswagen within the VG model shown in Figure 6.

but also the Samuelson model is only complete in theory. Since in reality just discrete hedging is feasible, one will encounter hedge errors within this framework, too. The magnitude of these errors is, of course, relevant for practical applications. In particular, one may ask if the hedge errors of the cost-efficient puts are comparable or even less than those of the standard puts because otherwise the advantage of the lower initial costs might be annihilated. Therefore, we finally calculate and compare the hedge errors that occur in delta hedging of the vanilla and efficient puts on Allianz and Volkswagen considered before.

We assume that the hedge portfolios are rebalanced daily, hence the portfolio weights δ_t (amount of stock at time t) and b_t (amount of money on the savings account at t) just have to be calculated at the discrete times $t = 0, 1, \ldots, T-1$. For the vanilla puts $\delta_t = \Delta^P_{T-t}$, and in case of the efficient puts we have $\delta_t = \underline{\Delta}^P_{T-t}$. Depending on the put type under consideration, we analogously set $c_t = c(X^P_{T-t})$ or $c_t = c(\underline{X}^P_{T-t})$, respectively.

At the initial time $t = 0$, the hedge portfolio is set up with the weights δ_0 and $b_0 = -\delta_0 S_0 + c_0$ since the writer of the put obtains c_0 from the buyer, shorts $|\delta_0|$ stocks and deposits all incomes on his savings account. At time $t > 0$, the value of the portfolio *before* rebalancing is $\delta_{t-1} S_t + e^r b_{t-1}$, and we define the corresponding hedge error by

$$e_t := c_t - \delta_{t-1} S_t - e^r b_{t-1},$$

so positive hedge errors mean losses and negative gains. At the end of the trading day, the new weights δ_t and $b_t = c_t - \delta_t S_t$ are chosen to ensure that the value of the portfolio again exactly coincides with the present put price. Using the above definition of e_t, we can alternatively represent b_t in the form

$$b_t = e_t + e^r b_{t-1} + S_t(\delta_{t-1} - \delta_t).$$

This means that the hedge error is nothing but the amount of money one has to additionally inject to or withdraw from the savings account after adapting the stock position to make the value of the hedge portfolio congruent with the current put price.

Remark 4.2 In general, the size of the hedge error also depends on the rebalancing frequency and the continuity properties of the payoff function. Our empirical results below show that for standard and efficient puts a daily rebalancing of the portfolio already is sufficient to get a fairly precise approximation to the current option prices. A thorough theoretical analysis of the behaviour of hedge errors resulting from delta and quadratic hedging strategies in exponential Lévy models can be found in Brodén and Tankov (2011).

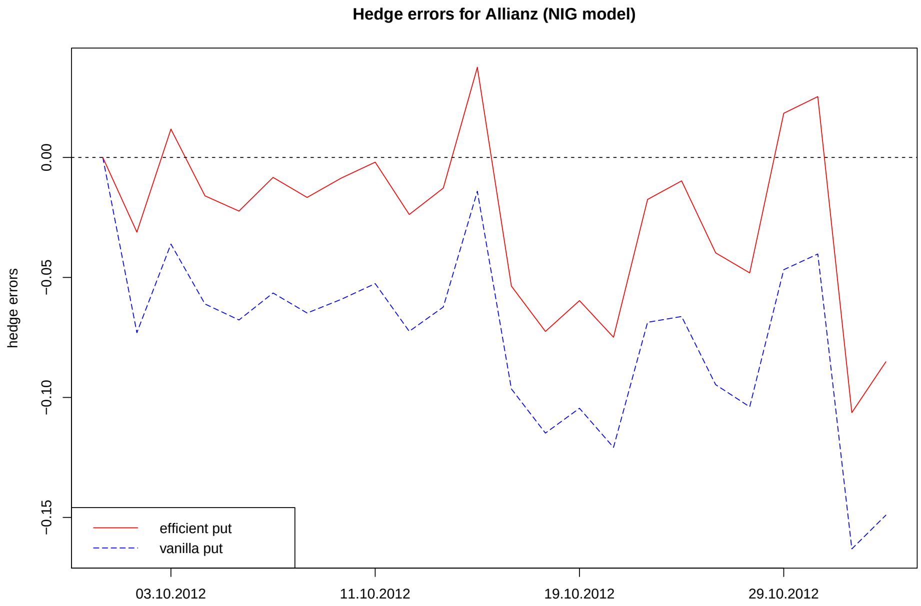

Figure 8: Delta hedge errors of the efficient and vanilla puts on Allianz with strike $K = 98$ maturing on November 1, 2012, in the NIG model.

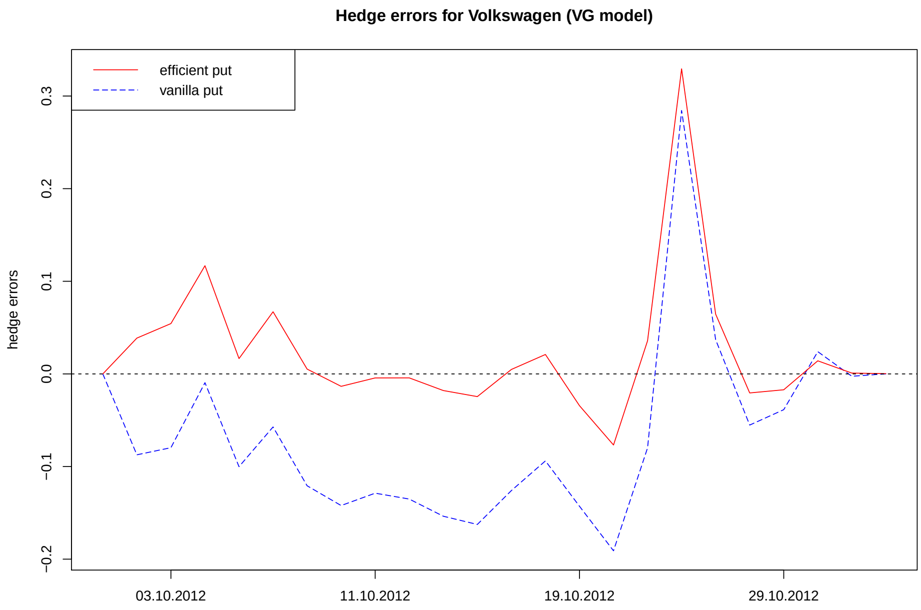

Figure 9: Delta hedge errors of the efficient and vanilla puts on Volkswagen with strike $K = 135$ maturing on November 1, 2012, in the VG model.

Figures 8 and 9 display the hedge errors obtained from delta hedging of the different puts on Allianz and Volkswagen. For Allianz, the absolute hedge errors $|e_t|$ of the efficient put are smaller than those of the vanilla put for all but one trading day. In case of Volkswagen, there are five days at which the absolute hedge errors of the efficient put are greater than those of the vanilla put, but still the sum of all absolute hedge errors is smaller for the efficient put. This indicates that delta hedging of cost-efficient options yields at least comparable and often even more accurate results than the corresponding hedging strategies for standard options as has already been expected from the proportions of the deltas derived in Theorem 4.2 (see also Remark 4.1). But since efficient puts are always cheaper than vanilla ones, one should not only look at the absolute hedge errors to confirm this assertion, but also take the relative or percentage hedge errors $\tilde{e}_t := \frac{e_t}{c_t}$ into account. The values of $\tilde{e}_t$ for the Allianz puts are shown in Figure 10 below. Analogous computations for the Volkswagen puts would not make much sense here because they end up deep out of the money, therefore the $\tilde{e}_t$ would tend to infinity as $t \to T$.

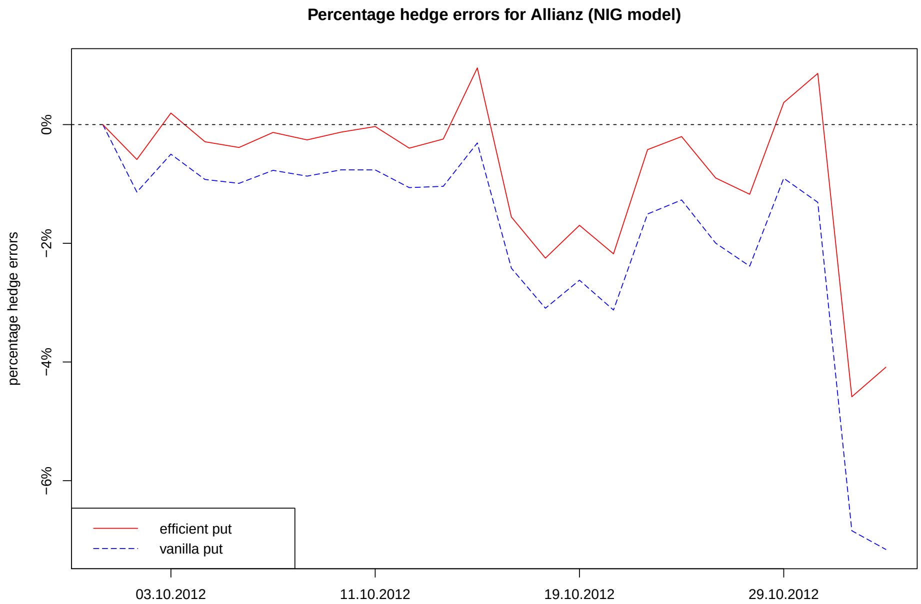

Figure 10: Percentage hedge errors of the efficient and vanilla puts on Allianz with strike $K = 98$ maturing on November 1, 2012, in the NIG model.

Clearly, the delta hedge of the efficient Allianz put is more accurate than that of the vanilla put. The latter tends to superhedge the option, that is, the value of hedge portfolio is always greater than the option price. Recalling that the payoffs of both puts must coincide at maturity, we see that cost-efficient puts and the corresponding delta hedges may indeed provide a much cheaper way to achieve and hedge a final payoff of the form $(K - S_T)_+$, and thus, can be regarded as a low-cost insurance against falling prices of the underlying. In view of Theorem 4.2, we suppose that analogous assertions will also hold for calls and probably for more complex options, too.

5 Conclusion

We applied the concept of cost-efficiency to general Lévy market models where the risk-neutral measure is obtained by an Esscher transform. Explicit criteria for cost-efficiency were derived and applied to various financial derivatives. We have also shown that the magnitude of the efficiency loss increases if the market trend, resp. the drift of the underlying, becomes more pronounced. Moreover, we established a cost-efficient version of the put-call parity. Numerical examples of cost-efficient puts were presented that evidenced that the savings from choosing the cost-efficient strategies can be quite large. We found that the efficiency losses obtained under different Lévy models were of almost the same magnitude, and thus, seem to be widely model-independent. Further, we derived explicit formulas for the Greek delta of cost-efficient puts and calls and proved that their absolute values are smaller than those of the corresponding vanilla option deltas. This suggests that delta hedging of efficient options is more accurate and leads to smaller hedge errors than that of standard options. In a practical application using German stock price data we demonstrated that the computation of the deltas is numerically tractable and that the associated hedging strategies for efficient puts indeed have the potential to outperform its counterparts for vanilla puts. This indicates that cost-efficient strategies provide a more advantageous way to achieve and hedge a final payoff, and thus, may be an appropriate tool to increase market efficiency.

A Derivation of the risk-neutral Esscher parameters

Normal inverse Gaussian model. Recall that by equation (2.9) the moment generating function of an NIG distribution is given by

$$M_{NIG(\alpha,\beta,\delta,\mu)}(u) = e^{u\mu+\delta\sqrt{\alpha^2-\beta^2}-\delta\sqrt{\alpha^2-(\beta+u)^2}},$$

so the defining equation (2.5) for the risk-neutral Esscher parameter here becomes

$$e^r = \frac{M_{NIG(\alpha,\beta,\delta,\mu)}(\bar{\theta}_{NIG}+1)}{M_{NIG(\alpha,\beta,\delta,\mu)}(\bar{\theta}_{NIG})} = e^{\mu-\delta\sqrt{\alpha^2-(\beta+\bar{\theta}_{NIG}+1)^2}+\delta\sqrt{\alpha^2-(\beta+\bar{\theta}_{NIG})^2}}$$

or equivalently

$$\frac{r-\mu}{\delta} = \sqrt{\alpha^2 - (\beta + \bar{\theta}_{NIG})^2} - \sqrt{\alpha^2 - (\beta + \bar{\theta}_{NIG} + 1)^2}. \tag{A.1}$$

Under Assumption ($\mathbb{M}$), which is equivalent to $2\alpha > 1$, Lemma 2.7 states that there can exist at most one solution $\bar{\theta}_{NIG}$ to (A.1) which obviously must also fulfill the additional constraints $|\beta + \bar{\theta}_{NIG}| < \alpha$ and $|\beta + \bar{\theta}_{NIG} + 1| < \alpha$.

Case 1: $r = \mu$

Here we have that

$$0 = \sqrt{\alpha^2 - (\beta + \bar{\theta}_{NIG})^2} - \sqrt{\alpha^2 - (\beta + \bar{\theta}_{NIG} + 1)^2}$$

and hence $(\beta + \bar{\theta}_{NIG})^2 = (\beta + \bar{\theta}_{NIG} + 1)^2$, which obviously is fulfilled iff

$$\bar{\theta}_{NIG} = -\frac{1}{2} - \beta.$$

This is a proper solution since by Assumption ($\mathbb{M}$) $\alpha > \frac{1}{2} = |\beta + \bar{\theta}_{NIG}| = |\beta + \bar{\theta}_{NIG} + 1|$.

Case 2: $r \neq \mu$

To simplify notations, we set $r^* = \frac{r-\mu}{\delta}$ and $\beta^* = \beta + \bar{\theta}_{NIG}$ in the following. Using these abbreviations, equation (A.1) can be rewritten as

$$\sqrt{\alpha^2 - \beta^{*2} - (2\beta^* + 1)} = \sqrt{\alpha^2 - \beta^{*2}} - r^*.$$

Squaring this equation and isolating the term $\sqrt{\alpha^2 - \beta^{*2}}$ yields

$$\sqrt{\alpha^2 - \beta^{*2}} = \frac{(1 + r^{*2}) + 2\beta^*}{2r^*}.$$

Squaring again and reorganizing terms we finally obtain the following quadratic equation for β^*:

$$\beta^{*2} + \beta^* + \frac{1 + r^{*2}}{4} - \frac{\alpha^2 r^{*2}}{1 + r^{*2}} = 0.$$

The solutions of this quadratic equation are given by

$$\beta^* = \beta + \bar{\theta}_{NIG} = -\frac{1}{2} \pm r^* \sqrt{\frac{\alpha^2}{1 + r^{*2}} - \frac{1}{4}} \Longrightarrow \bar{\theta}_{NIG} = -\frac{1}{2} - \beta \pm r^* \sqrt{\frac{\alpha^2}{1 + r^{*2}} - \frac{1}{4}}.$$

Note that the above solutions only exist if $2\alpha > \sqrt{1 + r^{*2}}$ which is more restrictive than Assumption ($\mathbb{M}$). From equation (A.1) we conclude that for $r^* > 0$ we must have $(\beta + \bar{\theta}_{NIG})^2 < (\beta + \bar{\theta}_{NIG} + 1)^2$, which is equivalent to $-\frac{1}{2} - \beta < \bar{\theta}_{NIG}$. If $r^* < 0$, we analogously arrive at the constraint $-\frac{1}{2} - \beta > \bar{\theta}_{NIG}$. Comparing this with the above solutions of the quadratic equation for β^*, we finally see that the only possible solution for the risk-neutral Esscher parameter is, re-inserting $\frac{r-\mu}{\delta} = r^*$,

$$\bar{\theta}_{NIG} = -\frac{1}{2} - \beta + \frac{r-\mu}{\delta}\sqrt{\frac{\alpha^2}{1+(\frac{r-\mu}{\delta})^2} - \frac{1}{4}}. \tag{A.2}$$

However, observe that this is a possible, but not a definitive solution! One additionally has to check if the obtained $\bar{\theta}_{NIG}$ really solves the initial equation (A.1). There exist sets of NIG parameters which fulfill all necessary constraints, however, the value $\bar{\theta}_{NIG}$ calculated according to (A.2) is not a valid solution of (A.1). Take, for example, $(\alpha, \beta, \delta, \mu) = (1, -0.1, 0.05, 0)$ and $r = 0.06$, then we have $2 = 2\alpha > \sqrt{1 + r^{*2}} = 1.56205$, and calculating $\bar{\theta}_{NIG}$ according to (A.2) yields $\bar{\theta}_{NIG} = 0.07975404$. Clearly, this $\bar{\theta}_{NIG}$ also fulfills the additional constraints $|\beta + \bar{\theta}_{NIG}| < \alpha$ and $|\beta + \bar{\theta}_{NIG} + 1| < \alpha$, but inserting this value and the other parameters into equation (A.1) one sees that the latter is violated.

Variance Gamma model. By equation (2.12), the moment generating function of a VG distribution is given by

$$M_{VG(\lambda,\alpha,\beta,\mu)}(u) = e^{u\mu}\left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + u)^2}\right)^{\lambda}.$$

Thus, the defining equation for the risk-neutral Esscher parameter in this case is

$$e^r = \frac{M_{VG(\lambda,\alpha,\beta,\mu)}(\bar{\theta}_{VG} + 1)}{M_{VG(\lambda,\alpha,\beta,\mu)}(\bar{\theta}_{VG})} = e^{\mu}\left(\frac{\alpha^2 - (\beta + \bar{\theta}_{VG})^2}{\alpha^2 - (\beta + \bar{\theta}_{VG} + 1)^2}\right)^{\lambda},$$

or equivalently,

$$e^{\frac{r-\mu}{\lambda}} = \frac{\alpha^2 - (\beta + \bar{\theta}_{VG})^2}{\alpha^2 - (\beta + \bar{\theta}_{VG} + 1)^2}. \tag{A.3}$$

As has alread been pointed out on p. 20, the condition $2\alpha > 1$ here is sufficient to guarantee the existence of a unique solution $\bar{\theta}_{VG}$ of equation (A.3).

Case 1: $r = \mu$

In this case, (A.3) becomes

$$\alpha^2 - (\beta + \bar{\theta}_{VG})^2 = \alpha^2 - (\beta + \bar{\theta}_{VG} + 1)^2$$

which apparently is solved by

$$\bar{\theta}_{VG} = -\frac{1}{2} - \beta.$$

This is a proper solution as can be seen analogously as in the NIG model.

Case 2: $r \neq \mu$

To simplify the notation and formulas in the derivation of $\bar{\theta}_{VG}$, we set, similarly as before, $r^* = \frac{r-\mu}{\lambda}$ and $\beta^* = \beta + \bar{\theta}_{VG}$. Multiplying both sides of (A.3) with $\alpha^2 - (\beta^* + 1)^2$ yields

$$e^{r^*}\left(\alpha^2 - (\beta^* + 1)^2\right) = \alpha^2 - \beta^{*2}.$$

Expanding the expressions and rearranging terms we obtain

$$\beta^{*2} + \frac{2}{1 - e^{-r^*}}\beta^* + \left(\frac{1}{1 - e^{-r^*}} - \alpha^2\right) = 0.$$

The solutions of this quadratic equation are given by

$$\beta^* = \beta + \bar{\theta}_{VG} = -\frac{1}{1 - e^{-r^*}} \pm \sqrt{\frac{e^{-r^*}}{(1 - e^{-r^*})^2} + \alpha^2}.$$

Thus, the possible risk-neutral Esscher parameters are

$$\bar{\theta}_{VG} = -\frac{1}{1 - e^{-r^*}} - \beta \pm \sqrt{\frac{e^{-r^*}}{(1 - e^{-r^*})^2} + \alpha^2}. \tag{A.4}$$

Observe that the moment generating function M_{VG} is only defined on the interval $(-\alpha - \beta, \alpha - \beta)$, therefore $\bar{\theta}_{VG} \in (-\alpha - \beta, \alpha - \beta - 1)$ must hold. Further, note that we always have $\sqrt{\frac{e^{-r^*}}{(1-e^{-r^*})^2} + \alpha^2} > \alpha$.

Now suppose that $r > \mu$, then $e^{-r^*} < 1$, or equivalently, $-\frac{1}{1-e^{-r^*}} < 0$, and thus

$$-\frac{1}{1 - e^{-r^*}} - \beta - \sqrt{\frac{e^{-r^*}}{(1 - e^{-r^*})^2} + \alpha^2} < -\beta - \alpha.$$

Hence, this solution of (A.3) does not lie within $(-\alpha - \beta, \alpha - \beta - 1)$, so the unique solution in the case $r > \mu$ is given by

$$\bar{\theta}_{VG} = -\frac{1}{1 - e^{-\frac{r-\mu}{\lambda}}} - \beta + \sqrt{\frac{e^{-\frac{r-\mu}{\lambda}}}{\left(1 - e^{-\frac{r-\mu}{\lambda}}\right)^2} + \alpha^2}. \tag{A.5}$$

If on the other hand $r < \mu$, then we have $e^{-r*} > 1$, resp. $-\frac{1}{1-e^{-r^*}} > 0$, so the above solution (A.5) lies outside $(-\alpha - \beta, \alpha - \beta - 1)$ because

$$-\frac{1}{1 - e^{-r^*}} - \beta + \sqrt{\frac{e^{-r^*}}{(1 - e^{-r^*})^2} + \alpha^2} > -\beta + \alpha.$$

Consequently, the unique solution in the case $r < \mu$ is given by

$$\bar{\theta}_{VG} = -\frac{1}{1 - e^{-\frac{r-\mu}{\lambda}}} - \beta - \sqrt{\frac{e^{-\frac{r-\mu}{\lambda}}}{\left(1 - e^{-\frac{r-\mu}{\lambda}}\right)^2} + \alpha^2}. \tag{A.6}$$

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